

Problems of the Month for UH Mānoa Undergraduates

Problems for June 2007

Easier Problem. Recall that a positive integer p is a prime number if and only if $p \neq 1$ and p has no positive integer divisors other than itself and 1. Find all positive integers n such that $2^n + 1$ and $2^n - 1$ are both prime numbers. Carefully justify your answer.

Harder Problem. For each positive integer n , let $z(n)$ be the number of zeros that n has when written in base 3. For example, 57 written in base 3 is 2010, since

$$57 = 2 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3^1 + 0 \cdot 3^0.$$

Thus $z(57) = 2$. Find all positive values of x , such that

$$\sum_{n=1}^{\infty} \frac{x^{z(n)}}{n^3} < \infty.$$

Hints. Note that n has $k+1$ digits in base 3 if $3^k \leq n \leq 3^{k+1} - 1$. Show that the number of such n that have $z(n) = i$ is $\binom{k}{i} 2^{k+1-i}$, where $\binom{k}{i} = \frac{k!}{i!(k-i)!}$. Explain why

$$\sum_{n=3^k}^{3^{k+1}-1} \frac{x^{z(n)}}{n^3} \leq 2 \frac{\sum_{i=0}^k \binom{k}{i} 2^{k-i} x^i}{(3^k)^3} = \frac{2}{3^{3k}} (2+x)^k.$$

Now sum from $k = 0$ to ∞ to get a function $g(x)$ (defined on a certain interval), such that $\sum_{n=1}^{\infty} \frac{x^{z(n)}}{n^3} \leq g(x)$. Similarly, find a function $f(x)$ such that $f(x) \leq \sum_{n=1}^{\infty} \frac{x^{z(n)}}{n^3}$.

Rules

The following rules may be changed or clarified from month to month.

1. Any “regular” undergraduate currently enrolled at UH Manoa is eligible to compete.

2. Write a complete solution with all details to either problem or both.
3. Submit your solution(s) *electronically* before the end of the above month to

bleecker@math.hawaii.edu

For the subject line of your email use “problem of the month”. Either write your solution within the body of your email or within attachment(s) in the form of readable pdf files or images of your work in jpg format (e.g., scanned or digitally photographed).

4. Solutions will be judged by a committee of professors according to a combination of criteria: accuracy, attention to details, chronological order of submission, and neatness, but not necessarily in that order.

5. In addition to recognition on the Department web site, winner(s) will receive a modest cash prize of a yet-to-be-determined amount from the William P. Hanf Memorial Fund* before a yet-to-be-determined time.

*Fondly remembered UH math professor, William P. Hanf is known for his contributions to mathematical logic. While at UC Berkeley, he scored among the top five contestants in the Putnam Mathematical Competition in 1954, along with Benjamin Muckenhoupt (Harvard), James Daniel Bjorken (MIT), and Leonard Evens (Cornell), and Kenneth G. Wilson (Harvard) who went on to win the Nobel Prize in Physics in 1982.