Solutions of Problems for June 2007

Easier Problem. Recall that a positive integer p is a prime number if and only if $p \neq 1$ and p has no positive integer divisors other than itself and 1. Find all positive integers n such that $2^n + 1$ and $2^n - 1$ are both prime numbers. Carefully justify your answer.

Harder Problem. For each positive integer n, let $z(n)$ be the number of zeros that n has when written in base 3. For example, 57 written in base 3 is 2010, since

$$57 = 2 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3^1 + 0 \cdot 3^0.$$

Thus $z(57) = 2$. Find all positive values of x, such that

$$\sum_{n=1}^{\infty} \frac{x^{z(n)}}{n^3} < \infty.$$

Hints. Note that n has $k+1$ digits in base 3 if $3^k \leq n \leq 3^{k+1} - 1$. Show that the number of such n that have $z(n) = i$ is $\binom{k}{i} 2^{k+1-i}$, where $\binom{k}{i} = \frac{k!}{i!(k-i)!}$. Explain why

$$\sum_{n=3^k}^{3^{k+1}-1} \frac{x^{z(n)}}{n^3} \leq 2 \frac{\sum_{i=0}^{k} \binom{k}{i} 2^{k-i} x^i}{(3^k)^3} = \frac{2}{3^{3k}} (2+x)^k.$$

Now sum from $k = 0$ to ∞ to get a function $g(x)$ (defined on a certain interval), such that $\sum_{n=1}^{\infty} \frac{x^{z(n)}}{n^3} \leq g(x)$. Similarly, find a function $f(x)$ such that $f(x) \leq \sum_{n=1}^{\infty} \frac{x^{z(n)}}{n^3}$.

Solution of Easier Problem. Since 2^n is not evenly divisible by 3, the remainder r of 2^n when divided by 3 is either 1 or 2. If $r = 1$, then $2^n - 1$ is divisible by 3. In that case, $2^n - 1 = 3$ since $2^n - 1$ is assumed to be a prime, and so the only possible value for n is 2 this case. Note that $2^2 + 1 = 5$ is also a prime and so $n = 2$ is such that $2^n - 1$, $2^n + 1$ are primes. If $r = 2$, then $2^n + 1$ is divisible by 3 and hence must be 3. However, then n would be 1, and $2^1 - 1 = 1$ is not a prime. Thus, 2 is the only value of n such that $2^n + 1$ and $2^n - 1$ are primes.

Harder Problem Solution. In constructing an integer number n in $[3^k, 3^{k+1}-1]$ with $k+1$ digits in base 3 for which which there are i zeros, we note that the first digit cannot be 0 and we select i of the remaining k digits to be 0. There are $\binom{k}{i} = \frac{k!}{i!(k-i)!}$ ways to choose these. There are two choices (1 or 2) for each of the remaining $k+1-i$ other digits. Hence, there are $\binom{k}{i}2^{k+1-i}$ integers n in $[3^k, 3^{k+1}-1]$ with $z(n) = i$. For those n, we have

$$\frac{x^i}{(3^{k+3})^3} \leq \frac{x^{z(n)}}{n^3} = \frac{x^i}{n^3} \leq \frac{x^i}{(3^k)^3},$$

since

$$3^k \leq n \leq 3^{k+1} - 1 \Rightarrow 3^k \leq n \leq 3^{k+1} \Rightarrow \frac{1}{(3^{k+1})^3} \leq \frac{1}{n^3} \leq \frac{1}{(3^k)^3}.$$

Thus, summing over all n in $[3^k, 3^{k+1}-1]$ with $z(n) = i$, we obtain

$$\frac{\binom{k}{i}2^{k+1-i}x^i}{(3^{k+3})^3} \leq \sum_{n \text{ in } [3^k, 3^{k+1}-1] \text{ and } z(n)=i} \frac{x^{z(n)}}{n^3} \leq \frac{\binom{k}{i}2^{k+1-i}x^i}{(3^k)^3}.$$

Summing over i from 0 to k, we get

$$\sum_{i=0}^{k} \frac{\binom{k}{i}2^{k+1-i}x^i}{(3^{k+1})^3} \leq \sum_{n=3^k}^{3^{k+1}-1} \frac{x^{z(n)}}{n^3} \leq \sum_{i=0}^{k} \frac{\binom{k}{i}2^{k+1-i}x^i}{(3^k)^3}.$$

Now,

$$\sum_{i=0}^{k} \frac{\binom{k}{i}2^{k+1-i}x^i}{(3^k)^3} = 2\frac{\sum_{i=0}^{k}\binom{k}{i}2^{k-i}x^i}{(3^k)^3} = 2\frac{(x+2)^k}{(3^3)^k} = 2\left(\frac{x+2}{27}\right)^k$$

and

$$\sum_{i=0}^{k} \frac{\binom{k}{i}2^{k+1-i}x^i}{(3^{k+1})^3} = 2\frac{\sum_{i=0}^{k}\binom{k}{i}2^{k-i}x^i}{(3^{k+1})^3} = 2\frac{(x+2)^k}{3^{3k+3}} = \frac{2}{27}\left(\frac{x+2}{27}\right)^k.$$

Thus,

$$\frac{2}{27}\left(\frac{x+2}{27}\right)^k \leq \sum_{n=3^k}^{3^{k+1}-1} \frac{x^{z(n)}}{n^3} \leq 2\left(\frac{x+2}{27}\right)^k$$

Summing from $k = 0$ to ∞, we get, provided $\frac{x+2}{27} < 1$ (i.e., $x \in (0, 25)$)

$$\frac{4}{25 - x} = \frac{2}{27} \frac{54}{25 - x} \leq \sum_{n=1}^{\infty} \frac{x^{z(n)}}{n^3} \leq 2 \frac{1}{1 - \frac{x+2}{27}} = \frac{54}{25 - x}$$

Thus, $\sum_{n=1}^{\infty} \frac{x^{z(n)}}{n^3} < \infty$ for $x \in (0, 25)$. If $x \geq 25$, then $\sum_{k=1}^{\infty} \frac{2}{27} \left(\frac{x+2}{27}\right)^k$ diverges and hence

$$\infty \leq \sum_{k=1}^{\infty} \frac{2}{27} \left(\frac{x+2}{27}\right)^k \leq \sum_{n=1}^{\infty} \frac{x^{z(n)}}{n^3},$$

namely $\sum_{n=1}^{\infty} \frac{x^{z(n)}}{n^3} = \infty$.