Problems of the Month for UH Mānoa Undergraduates

Solutions for March/April 2008

Problem A. Let $x_1 = 1$, and for an integer $n \geq 1$, let

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right),$$

so that $x_2 = \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) = 3/2$, $x_3 = \frac{1}{2}\left(x_2 + \frac{2}{x_2}\right) = \frac{1}{2}\left(3/2 + \frac{2}{3/2}\right) = \frac{17}{12}$, etc.

(a) Prove that

$$\lim_{n\to\infty} x_n = \sqrt{2},$$

i.e., for any $\varepsilon > 0$, there is some integer $N > 0$, such that $\left|x_n - \sqrt{2}\right| < \varepsilon$ for $n > N$.

(b) Assume now that $x_1 = -1$ and we still have $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$. Now find $\lim_{n\to\infty} x_n$ and prove that your answer is correct.

Problem B. Suppose that every point in the xy-plane is colored red, yellow or blue. Show that there are 2 points of the same color which are exactly 1 unit apart.

Solution of Problem A. (a) Let $y_n = x_n - \sqrt{2}$. We need to show that $\lim_{n\to\infty} y_n = 0$. We have

$$\begin{aligned}
y_{n+1} = x_{n+1} - \sqrt{2} &= \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) - \sqrt{2} \\
&= \frac{1}{2}\left(\left(y_n + \sqrt{2}\right) + \frac{2}{\left(y_n + \sqrt{2}\right)}\right) - \sqrt{2} \\
&= \frac{1}{2}\left(\frac{y_n^2 + 2y_n\sqrt{2} + 2 + 2}{y_n + \sqrt{2}}\right) - \frac{\sqrt{2}\left(y_n + \sqrt{2}\right)}{y_n + \sqrt{2}} \\
&= \frac{1}{2}\frac{y_n^2}{y_n + \sqrt{2}} = \frac{1}{2}\left(\frac{y_n}{y_n + \sqrt{2}}\right) y_n
\end{aligned}$$

Since $y_2 = x_2 - \sqrt{2} = 3/2 - \sqrt{2} > 0$, it follows by induction that $y_n > 0$ for all $n \geq 1$, and for $k \in \{0, \ldots, n-1\}$ and $n \geq 2$

$$y_{n+1} = \frac{1}{2}\left(\frac{y_n}{y_n + \sqrt{2}}\right) y_n \leq \frac{1}{2} y_n \leq \frac{1}{2^2} y_{n-1} \leq \frac{1}{2^{k+1}} y_{n-k} \leq \cdots \leq \frac{1}{2^{n-2}} y_2.$$

Thus,

$$0 \leq \lim_{n\to\infty} y_n = \left(\lim_{n\to\infty} \frac{1}{2^{n-2}}\right) y_2 = 0.$$

(b) Let $z_1 = -1$ and $z_{n+1} = \frac{1}{2}\left(z_n + \frac{2}{z_n}\right)$. Then

$$z_2 = \frac{1}{2}\left(z_1 + \frac{2}{z_1}\right) = \frac{1}{2}\left(-1 + \frac{2}{-1}\right) = -\frac{3}{2} = -x_1$$
$$z_3 = \frac{1}{2}\left(z_2 + \frac{2}{z_2}\right) = \frac{1}{2}\left(-\frac{3}{2} + \frac{2}{-\frac{3}{2}}\right) = -\frac{17}{12} = -x_2$$

Suppose that $z_n = -x_n$. Then

$$z_{n+1} = \frac{1}{2}\left(z_n + \frac{2}{z_n}\right) = \frac{1}{2}\left(-x_n + \frac{2}{-x_n}\right) = -\frac{1}{2}\left(x_n + \frac{2}{x_n}\right) = -x_{n+1}.$$

Hence by induction, $z_n = -x_n$ for all n. Thus,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(-x_n) = -\lim_{n\to\infty} x_n = -\sqrt{2}.$$

Solution of Problem B. Consider two equilateral triangles in the plane with sides of length 1 sharing a common side P_2P_3, say $P_1P_2P_3$ and $P_2P_3P_4$. The colors of P_2 and P_3 must be different or else we are done. P_1 and P_4 must both have the remaining color. Rotate the rhombus $P_1P_2P_4P_3$ about P_1. Then P_4 traces out a monochromatic circle C of radius larger than 1 (indeed $\sqrt{3}$) about P_1. There are many pairs of points on C separated by distance 1.