

# Problems of the Month for UH Mānoa Undergraduates

## Problems for November 2007

**Problem A.** Note that  $\sqrt{2}$  is not the quotient  $\frac{m}{n}$  of two integers  $m$  and  $n$ . Indeed,  $\sqrt{2} = \frac{m}{n}$  implies that  $2n^2 = m^2$  which implies that  $m$  has a factor of 2, say  $m = 2m'$  for some integer  $m'$ . Then  $2n^2 = m^2 = 4(m')^2$  and  $n^2 = 2(m')^2$ , in which case  $n$  has a factor of 2, say  $n = 2n'$ . Then  $\frac{m}{n} = \frac{m'}{n'}$ , and continuing the argument,  $m$  and  $n$  have infinitely many factors of 2, which is impossible. Show that there are two real numbers  $x$  and  $y$  (possibly equal) neither of which is a quotient of two integers, such that  $x^y$  is a quotient of two integers. **Hint.** You need not find values for  $x$  and  $y$ . Just provide a short proof that such numbers exist. There is a proof which is only a few lines long!

**Problem B.** Recall that  $2! = 2$ ,  $3! = 3 \cdot 2 = 6$ , and for any integer  $n > 3$ ,  $n! = n \cdot (n-1)!$ . For  $n \geq 4$ , let  $P_n(x)$  be the polynomial defined by

$$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n.$$

Show that for all  $n > 4$ , there is no polynomial  $Q_n(x)$  and real number  $a_n$ , such that

$$P_n(x) = (x - a_n)^2 Q_n(x).$$

## Rules

The following rules may be changed or clarified from month to month.

1. Any “regular” undergraduate currently enrolled at UH Manoa is eligible to compete.
2. Write a complete solution with all details to either problem or both.
3. Submit your solution(s) *electronically* before the end of the above month to

bleecker@math.hawaii.edu

For the subject line of your email use “problem of the month” and *send your email via your UH email address*. Either write your solution within the body of your email or within attachment(s) in the form of readable pdf files or images of your work in jpg format (e.g., scanned or digitally photographed).

4. Solutions will be judged by a committee of professors according to a combination of criteria: accuracy, attention to details, chronological order of submission, and neatness, but not necessarily in that order.

5. Before the end of the 10-th day of the month that follows, the winner(s) will be announced on the Math Department web site. Moreover, if there is at least one good answer to a problem the winner(s) for that problem will collectively receive a total of least \$20 to be distributed among them depending on the criteria in 4 above, as soon as the checks can be extracted from the Hanf Fund at the UH Foundation, a process that may take several weeks.