Problems of the Month for UH Mānoa Undergraduates
Solutions for October 2007

Problem A. Let $f(x)$ be a function defined and continuous at all real numbers x. Suppose that there is no real number r such that $f(r) = r$. Prove that there is also no real number r such that $f(f(r)) = r$. You may use any theorem stated in a calculus book.

Problem B. Let $(\cos q_0, \sin q_0) = (p_1, q_1) = (4/5, 3/5)$. For $n \geq 0$, suppose that

$$\begin{aligned} p_{n+1} &= p_n \cos q_n - q_n \sin q_n \\ q_{n+1} &= p_n \sin q_n + q_n \cos q_n \end{aligned}$$

Find the limits $\lim_{n\to\infty} p_n$ and $\lim_{n\to\infty} q_n$ and prove that your answer is correct. **Hint**. Consider the sequence $\{\theta_n\}$ given by $\theta_n = q_0 + \cdots + q_n$. Show that $p_{n+1} = \cos(\theta_n)$ and $q_{n+1} = \sin(\theta_n)$, and that $\{\theta_n\}$ is an increasing sequence bounded above by π.

Solution of Problem A. Let $g(x) = f(x) - x$. We first prove that $g(x) > 0$ for all x or $g(x) < 0$ for all x. If not, then there are x_1 and $x_2 \in \mathbb{R}$ such that $g(x_1) \leq 0$ and $g(x_2) \geq 0$. Since $g(x_1) \neq 0$ by the assumption $f(x_1) \neq x_1$ and similarly $g(x_2) \neq 0$, we have $g(x_1) < 0$ and $g(x_2) > 0$. Since g is continuous, by the intermediate value theorem, $g(x_3) = 0$ for some x_3 between x_1 and x_2, yielding the contradiction $f(x_3) = x_3$. Thus, $g(x) > 0$ for all x or $g(x) < 0$ for all x. If $g(x) > 0$ for all x, then $f(x) > x$ for all x, in which case

$$f(f(x)) > f(x) > x,$$

while $g(x) < 0$ for all x implies

$$f(f(x)) < f(x) < x.$$

Solution of Problem B. Let $\cos q_0 = 4/5 = p_1$ and $\sin q_0 = 3/5 = q_1$. Then

$$\begin{aligned} p_2 &= p_1 \cos q_1 - q_1 \sin q_1 = \cos q_0 \cos q_1 - \sin q_0 \sin q_1 = \cos(q_0 + q_1) \\ q_2 &= p_1 \sin q_1 + q_1 \cos q_1 = \cos q_0 \sin q_1 + \sin q_0 \cos q_1 = \sin(q_0 + q_1). \end{aligned}$$

Having established the base case, we prove by induction that

$$p_n = \cos(q_0 + \cdots + q_{n-1}) \text{ and } q_n = \sin(q_0 + \cdots + q_{n-1}).$$

Indeed, assuming this

$$\begin{aligned} q_{n+1} &= \cos(q_0 + \cdots + q_{n-1}) \sin q_n + \sin(q_0 + \cdots + q_{n-1}) \cos q_n \\ &= \sin(q_0 + \cdots + q_n) = \sin\theta_n, \end{aligned}$$

where $\theta_n := q_0 + \cdots + q_n$. Moreover,

$$\begin{aligned} p_{n+1} &= p_n \cos q_n - q_n \sin q_n \\ &= \cos(q_0 + \cdots + q_{n-1}) \cos q_n - \sin(q_0 + \cdots + q_{n-1}) \sin q_n \\ &= \cos(q_0 + \cdots + q_n) = \cos\theta_n \end{aligned}$$

We claim

$$\theta_n \leq \theta_{n+1} < \pi$$

We have this for $n = 0$ (i.e., $\theta_0 \leq \theta_1 < \pi$), since

$$\theta_0 = q_0 = \cos^{-1}(4/5) \leq \cos^{-1}(4/5) + 3/5 = q_0 + q_1 = \theta_1$$

and

$$\theta_1 = \cos^{-1}(4/5) + 3/5 \approx 1.243\,501\,109 \leq \pi.$$

Thus,

$$\theta_0 \leq \theta_1 < \pi$$

We use induction. Assume $\theta_{n-1} \leq \theta_n \leq \pi$. Then

$$\begin{aligned} \theta_{n-1} \leq \theta_n \leq \pi &\Rightarrow q_{n+1} = \sin\theta_n \geq 0 \\ &\Rightarrow \theta_{n+1} = q_0 + \cdots + q_n + q_{n+1} \geq q_0 + \cdots + q_n = \theta_n \end{aligned}$$

and

$$\begin{aligned} q_{n+1} &= \sin\theta_n = \sin(\pi - \theta_n) \leq \pi - \theta_n = \pi - (q_0 + \cdots + q_n) \\ &\Rightarrow \theta_{n+1} = q_0 + \cdots + q_n + q_{n+1} \leq \pi \Rightarrow \theta_{n+1} \leq \pi. \end{aligned}$$

Thus,

$$\theta_{n-1} \leq \theta_n \leq \pi \Rightarrow \theta_n \leq \theta_{n+1} \leq \pi.$$

Since $\{\theta_n\}$ is a monotonically increasing sequence bounded above by π, it has a limit $L := \lim_{n\to\infty} \theta_n \leq \pi$. Moreover, since $\theta_n = q_0 + \cdots + q_n$, it follows that $\lim_{n\to\infty} q_n = 0$. Then

$$0 = \lim_{n\to\infty} q_{n+1} = \lim_{n\to\infty} \sin\theta_n = \sin\left(\lim_{n\to\infty} \theta_n\right) = \sin\left(L\right),$$

and so $L = \pi$. Finally,

$$\lim_{n\to\infty} p_{n+1} = \lim_{n\to\infty} \cos\theta_n = \cos\left(\lim_{n\to\infty} \theta_n\right) = \cos\left(L\right) = \cos\left(\pi\right) = -1.$$