Hanf Problem (September, 2008)

Problem. *Let* $p(x) = \sum_{k=0}^{n} a_k x^k$, *denote a non-constant polynomial, where the coefficients satisfy*

$$0 < a_0 < a_1 < \cdots < a_n. \tag{1}$$

(a) *Prove that any real zero of* $p(x)$ *lies in the open interval* $(-1, 0)$.

(b) *Show that*

$$\frac{1}{2\pi i}\int_C \frac{p'(w)}{p(w)}\,dw = n, \tag{2}$$

where C *is the positively oriented unit circle* ($|w| = 1$) *traversed exactly once.*

Proof. (a) Since the coefficients of $p(x)$ are positive, $p(x)$ cannot vanish for non-negative values. Now set $q(x) := x^n p(\frac{1}{x}) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ and note that (since $a_0 a_n \neq 0$) x_0 is a (real) zero of $p(x)$ if and only if $\frac{1}{x_0}$ is a (real) zero of $q(x)$. We will show that $q(x) > 0$ for $|x| \leq 1$. To this end, we fix x, where $|x| \leq 1$ and $x \neq 1$. Then

$$\begin{aligned}
&|(1-x)(a_0 x^n + a_1 x^{n-1} + \cdots + a_0)| = |a_n - (a_n - a_{n-1})x - \cdots - (a_1 - a_0)x^n - a_0 x^{n+1}| \\
&\geq a_n - |(a_n - a_{n-1})x + \cdots + (a_1 - a_0)x^n + a_0 x^{n+1}| \\
&> a_n - (a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_1 - a_0 + a_0) = a_n - a_n = 0,
\end{aligned}$$

where the *strict inequality* is a consequence of the fact that the terms $(a_n - a_{n-1})x, (a_{n-1} - a_n)x^2, \cdots, (a_1 - a_0)x^n, a_0 x^{n+1}$ cannot all have the same sign, unless they are all positive (in which case $q(x) > 0$). Thus, any (real) zero of $q(x)$ has modulus strictly greater than 1 and whence the real zeros of $p(x)$ lie in the open interval $(-1, 0)$.

(b) The first step of the proof of part (b) is, *mutatis mutandis*, the same as in part (a). We fix a complex number z, where $|z| \leq 1$ and $z \neq 1$. Then, as in part (a), the *strict inequality* is a consequence of the fact that $(a_n - a_{n-1})z, (a_{n-1} - a_n)z^2, \cdots, (a_1 - a_0)z^n, a_0 z^{n+1}$ cannot all have the same argument, unless they are all positive. Hence it follows that all the zeros of $q(z)$ have modulus strictly greater than 1 and whence all the zeros of $p(z)$ have modulus strictly less than 1. Thus, by the Argument Principle, (2) holds. □