Problems of the Month for UH Mānoa Undergraduates

Problems for November 2008

Problem A.
Let $f(x)$ be a function defined on the set of all real numbers. If there is a constant $k > 0$ such that

$$f(x+k)\,(1-f(x)) = 1 + f(x)$$

for all x, show that f is periodic.

Solution.
We will show that $f(x+4k) = f(x)$ for all x. Indeed, since $f(x) \neq 1$,

$$f(x+k) = \frac{1+f(x)}{1-f(x)}.$$

Hence, it follows that

$$f(x+2k) = -\frac{1}{f(x)},$$

and whence

$$f(x+4k) = -\frac{1}{f(x+2k)} = f(x).$$

□

Problem B.
Evaluate the definite integral

$$\int_0^{\pi/2} \left(\sin^2(\sin(x)) + \cos^2(\cos(x))\right)\, dx.$$

Answer: $\pi/2$

Solution.

$$\begin{aligned}
&\int_0^{\pi/2} \left(\sin^2(\sin(x)) + \cos^2(\cos(x))\right)\, dx \\
&= \int_0^{\pi/2} \left(\sin^2(\sin(x)) + 1 - \sin^2(\cos(x))\right)\, dx \\
&= \frac{\pi}{2} + \int_0^{\pi/2} \left(\sin^2(\sin(x)) - \sin^2(\cos(x))\right)\, dx.
\end{aligned}$$

The latter integral is equal to zero, since $\cos(x) = \sin(\pi/2 - x)$. □