Practice Problems

The problems are roughly grouped by the ideas required for their solutions. There may be, however, several ideas involved in the solution of a single problem. In every group, problems are listed, roughly, in order of increasing difficulty.

The pigeonhole principle.

1 Consider $A = 8$ natural numbers not exceeding $B = 15$. For every pair of these numbers calculate the absolute value of the difference of the numbers in the pair. Prove that there exist $C = 3$ pairs such that the absolute values of the differences of the numbers in the pairs are equal.

Try to generalize for other values of A, B, and C.

2 There are 289 points situated in the interior of a square of a side length of 1 foot. Prove that there is a square with a side length of 1 inch which contains at least 3 of these points.

3 A square of size 30×30 is tiled with squares of size 1×1 of four colors: red, green, pink and blue; the squares are colored such that no two squares of the same color touch each other (i.e., no side or vertex belongs to two squares of the same color). How many green squares are there?

4 On a plane, every point which has integer coordinates is assigned one out of n colors (here n is a positive integer.) Prove that there exists a rectangle with vertices at points with integer coordinates such that all of its vertices are the same color.

5 Consider a hundred integers. Prove that one can pick several of them (maybe only one) such that their sum is divisible by 100.

6 Prove that for any odd integer a there exists an integer b such that $2^b - 1$ is divisible by a.

7 Consider an arbitrary convex quadrilateral. For every side construct a circle which has the side as its diameter. Prove that these fuor circles together cover the quadrilateral.

Choosing an extremal element in a finite set

1 There are several natural numbers arranged in a circle. Every one of them equals the arithmetic mean of its two neighbors. Prove that all the numbers are equal.

2 There are several points in the plane such that the distances between pairs of them are pairwise different. Every point is connected with an interval to the closest one. Can this set of intervals represent a closed loop?

3 Prove that the equation

$x^x + y^y = z^z + w^w$

has no solution such that x, y, z, and w are pairwise different natural numbers.

4 Is it possible to situate several intervals on the plane such that every endpoint of every interval belongs to the interior of another interval?

5 Prove that a circle of radius 1 does not contain 6 points such that all the distances between them are strictly bigger than 1.

6 The sum of a hundred different positive integers equals 5051. Prove that this condition determines the numbers uniquely, and find them.

7 Consider 101 natural numbers not exceeding 200. Prove that among these numbers there must be two such that one of them is divisible by another one.

Invariants

1 There are 20 numbers: $1, 2, 3, \ldots, 20$ written on a blackboard. A move constitutes erasing any two, say, a and b, of them and writing the number $a + b - 1$ instead. After 19 moves, one number is left on the blackboard. What can you say about this number?

2 Consider the set of integers $1, 2, 3, \ldots, 1001$. It is allowed to add 1 to any two of these numbers in one move. Is it possible to make all the numbers equal after several moves?

3 A matrix (a rectangular table of numbers) of size $m \times n$ has the property that the sum of all numbers in every column and every row is 1. Prove that $m = n$.

4 There are three heaps of stones, which consist of 51, 49, and 5 stones correspondingly. Two operations are allowed: one can join any two heaps, and one can separate a heap which consists of an even number of stones into two equal heaps. Is it possible to obtain 105 $(= 51 + 49 + 5)$ heaps out of one stone each in this way.

Helpful coloring

1 There are 13 bricks of size $1 \times 1 \times 2$.Is it possible to construct out of them a cube of size $3 \times 3 \times 3$with a hole of size $1 \times 1 \times 1$ in the middle of it?

2 There are n arbitrary cells marked on a sheet of quad paper (paper with a grid drawn on it). Prove that there exist $n/4$ marked cells which have no common points pairwise (neither vertices nor sides).

3 On a sheet of quad paper (paper with a grid drawn), there is a convex n-gone such that all its vertices are vertices of the grid. Given that there are no vertices of the grid neither inside the n-gone, nor on its sides, prove that $n \leq 4$.

4 The plane is painted in three colors (i.e. every point of the plane is assigned one of the three colors). Prove that there exist two points of the same color such that the distance between them is 1.

5 Every side of an isosceles triangle is divided into n equal intervals. The lines parallel to the sides of the triangle split the triangle into small triangles. A sequence of small triangles such that the neighbor elements have a common side, and neither element appears twice, is called a chain. What is the maximum possible length of a chain?

6 A rectangle is tiled with rectangles of sizes 2×2 and 1×4. Prove that if one uses the same set of rectangles but with one 2×2 rectangle substituted by one 1×4 rectangle, the tiling becomes impossible.

7 Is it possible to tile a piece of quad paper(paper with a grid drawn on it) of size 29×29 with rectangles of size 1×4?

Induction arguments

1 Let n be a positive integer. Prove that among $n+1$ positive integers which do not exceed $2n$ there are two numbers such that one is divisible by another one.

2 A collection of n natural numbers $a_1 \ldots a_n$ has the property $a_k \leq k$ for every $k \leq n$. Prove that
$a_1 \pm a_2 \pm a_3 \ldots \pm a_n \neq 0$
for any choice of the signs $+$ and $-$.

3 Prove that
$n^n > (n+1)^{(n-1)}$
for any natural number n.

Miscellaneous

1 Find all integral solutions (i.e. x, y, and z must be integers) of the equation
$x^3 - 2y^3 - 4z^3 = 0$

2 There are 2000 apples in one or several baskets. It is allowed to take apples out of a basket, and to remove baskets. Prove that one can end up with a configuration such that all the baskets have equal amounts of apples, and the total amount of apples is not smaller than 100.

3 There are 20 different positive integers not exceeding 69. Prove that there are four equal pairwise differences of these numbers.

4 Is it possible for the product of two consecutive positive integers to be equal to the product of two consecutive odd numbers?

5 A sequence of numbers x_n is given by the following initial terms
$x_1 = 19 \qquad x_2 = 97$
and a recurrence relation
$x_{n+2} = x_n - \frac{1}{x_{n+1}}$
Find m such that $x_m = 0$.