Fatou and Julia Sets of Quadratic Polynomials

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Fatou and Julia sets of rational functions arise from the iteration of rational, complex valued functions. The boundaries of these sets create the intricate and often beautiful fractal images that were first generated by computers mathematically in the 1980's but have always existed in nature. As part of the M.A. Plan B degree requirements for the Mathematics Department at the University of Hawaii, this presentation will describe the Fatou and Julia sets for any quadratic polynomial in terms of the connectedness of the sets. I begin by defining the language used to discuss the iteration of a rational function and then discuss two theorems which characterize the Julia set as connected or totally disconnected. Finally, I present a theorem which defines the Mandelbrot set based on the connectedness of the Julia sets for any quadratic polynomial.

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, the Riemann Sphere. Then $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$, $R = \frac{P}{Q}$ is a **rational map** for P, Q coprime, not-both zero polynomials with $deg(R) = \max\{deg(P), deg(Q)\}$. Next define the $\mathbf{n^{th}}$ **iterate** R^n of R by $R^2 = R \circ R$ and $R^n = R^{n-1} \circ R$. Notice that $deg(R^n) = deg(R)^n$ for $deg(R) \geq 1$. Now, the **Fatou Set** of R, denoted $F(R)$, is defined as the set of points z_o in $\mathbb{C}_\infty$ such that the family of iterates R^n is a normal family in some neighborhood N_{z_o} of z_o. That is, every infinite sequence of R^n contains a subsequence R^{n_k} that converges locally uniformly to some $f \in \mathcal{C}(\mathbb{C}_\infty)$ on N_{z_o}. Recall that $R^{n_k} \to f$ locally uniformly on N_{z_o} if $\forall w \in N_{z_o}$, $R^n_k \to f$ uniformly on some neighborhood of w. Note that $f(z) \equiv \infty$ is considered an analytic function in $\mathcal{C}(\mathbb{C}_\infty)$ with the chordal metric, σ. Thus the rational maps are the analytic functions on $\mathbb{C}_\infty$. The **Julia Set**, $J(R)$, is the complement of the Fatou set, $J(R) = \mathbb{C}_\infty - F(R)$.

The Julia set, (and consequently the Fatou set) of two simple quadratic polynomials is easily calculated. These two examples, $P(z) = z^2$ and $P(z) =$

$z^2 - 2$, will be used to illustrate some of the elementary properties of rational maps.

First define two rational maps S and R to be **conjugate** if there exists g, a Möbius map such that $S = g \circ R \circ g^{-1}$. Conjugacy is an equivalence relation and respects degree and iteration, that is, $deg(R) = deg(S)$ and $S^n = g \circ R^n \circ g^{-1}$. The following theorem relates the Fatou and Julia sets of a polynomial to the Fatou and Julia sets of its conjugates.

Theorem 1 *If R is a nonconstant rational map, and g is a Möbius map such that $S = g \circ R \circ g^{-1}$, then $F(S) = g(F(R))$ and $J(S) = g(J(R))$.*

This result follows since the Möbius map g satisfies a Lipshitz condition with respect to the chordal metric on $\mathbb{C}_\infty$ so if $z_o \in F(R)$, then $g(z_o) \in F(S)$, and $g(F(R)) \subset F(S)$.

Now g^{-1} is a Möbius map and $R = g^{-1} \circ S \circ g$. Applying the above result gives $g^{-1}(F(S)) \subset F(R) \Rightarrow F(S) \subset g(F(R))$.

Example 2 *The Julia set for $P(z) = z^2$ is the unit circle.*

The discussion of this example is organized into three claims about the Fatou set of $P(z)$. Let $D(a, b) = \{z \in \mathbb{C} : |z - a| < b\}$ where $a \in \mathbb{C}$ and $0 < b \in \mathbb{R}$.

Claim 1: $z_o \in D(0, 1) \Rightarrow z_o \in F(P)$.

Let $z_o \in D(0, 1)$. Consider $U = D(z_o, \frac{1-|z_o|}{2}) \subset \overline{U} \subset D(0, 1)$. Now $\forall z \in \overline{U}, |P^n(z)| = |z|^{2^n} \leq \left(\frac{|z_o|+1}{2}\right)^{2^n}$. Now since $\left(\frac{|z_o|+1}{2}\right)^{2^n} \to 0$ as $n \to \infty$, $P^n(z) \to 0$ (the function that is identically equal to zero) uniformly on $\overline{U}$. So $\{P^n\}$ is normal in $D(0, 1)$, thus $D(0, 1) \subset F(P)$.

Claim 2:$|z_o| > 1 \Rightarrow z_o \in F(P)$.

Now let $|z_o| > 1$. Consider $S = g \circ P \circ g^{-1}$, with $g(z) = \frac{1}{z}$. $|g(z_o)| < 1$ implies that $g(z_o) \in F(z^2)$ by Claim 1. And since $S(z) = z^2 = P(z)$, $F(z^2) = g(F(z^2))$ by Theorem 1, $z_o \in F(P)$. Also notice that since $|z_o| > 1, |P^n(z_o)| = |z_o|^{2^n} \to \infty$ as $n \to \infty$.

Claim 3: $|z_o| = 1 \Rightarrow z_o \notin F(P)$.

Let $|z_o| = 1$. Assume $z_o \in F(P)$ so there is a neighborhood N, a subsequence of $\{P^n\}$, and a function f, with $\{P^{n_k}\} \to f$ locally uniformly on

N. Now $\forall \varepsilon > 0$ such that $D(z_o, \varepsilon) \subset N$, there exists $z_1 \in D(z_o, \varepsilon)$ with $|z_1| < 1$. So by Claim 1, $f(z_1) = 0$. But $|f(z_o)| = 1$. This contradicts that f is analytic (and therefore continuous).

Next define a set $E \subset X$ to be **completely invariant** under a function $g : X \to X$ if $g(E) = E = g^{-1}(E)$. Several important results follow:

- If E is completely invariant under g and $h : X \to X$ is a bijection, then $h(E)$is completely invariant under hgh^{-1}.
- If E is completely invariant under g, then E is completely invariant under g^n.
- If $g : X \to X$ is a continuous open map and $E \subset X$ is completely invariant under g then so are the complement $X - E$, the interior E^o, the closure $\overline{E}$, and the boundary ∂E of E. (Beardon Theorem 3.2.3 p. 53)
- The Fatou and Julia sets of a rational map R with $deg(R) \geq 2$ are completely invariant under R. (Beardon Theorem 3.2.4 p. 54)

Now consider $z_o \in \mathbb{C}_\infty$ such that $R(z_o) = z_o$. Then z_o is a **fixed point of R**. The derivative of R evaluated at z_o, $R'(z_o) = \lambda$, is called the **multiplier of R at $\mathbf{z_o}$**. The fixed point is classified based on the value of $|\lambda|$. If $|\lambda| > 1$, then z_o is a repelling fixed point. If $|\lambda| < 1$, then z_o is an **attracting fixed point**. z_o is a **superattracting fixed point** if $|\lambda| = 0$. Suppose z_o is an attracting fixed point of R and $|\lambda| < \rho < 1$. Then $|R(z) - R(z_o)| = |R(z) - z_o| \leq \rho|z - z_o|$ on some neighborhood D of z_o. Then $|R^n(z) - R^n(z_o)| = |R^n(z) - z_o| \leq \rho^n|z - z_o|$ and $R^n \to z_o$ uniformly on D. Now define the **basin of attraction of $\mathbf{z_o}$**, $A(z_o) = \{z \in \mathbb{C}_\infty : R^n(z) \to z_o\}$. This terminology allows for another definition of the Julia set.

Theorem 3 *For $P(z)$ with $deg(P) \geq 2$, the boundary of $A(\infty)$ coincides with the Julia set of P.*

Before giving an indication of the proof of Theorem 3, we show that it can be used to determine two simple Julia sets. Notice that the result of Example 2 follows immediately since the basin of attraction for $P(z) = z^2$ is the exterior of the unit disk.

Example 4 *The Julia set for $P(z) = z^2 - 2$ is the line segment $[-2, 2]$.*

Claim 1: The set $[-2,2]$ is completely invariant under $P(z)$.

> Consider $P(x) = x^2 - 2$. $P'(x) = 2x$ and $P''(x) = 2$ so $P(x)$ has a minimum value of -2 at $x = 0$. $P(x)$ is also an even function which increases on the interval $[0,2]$. So $P(x)$ has a maximum value of 2 at $x = \pm 2$. Thus $P([-2,2]) \subset [-2,2]$ and $[-2,2]$ is not a subset of $A(\infty)$.

Claim 2: $\Omega = \mathbb{C}_\infty - [-2,2]$ is $A(\infty)$.

> Let $h(\zeta) = \zeta + \frac{1}{\zeta}$ which maps $\{|\zeta| > 1\}$ onto $\mathbb{C}_\infty \backslash [-2,2]$. $h^{-1} \circ P \circ h = \zeta^2$ so $P(z)$ is conjugate to ζ^2. Since the iterates of any ζ under ζ^2 tend to ∞ for $\{|\zeta| > 1\}$, the iterates of $z \in \mathbb{C}_\infty \backslash [-2,2]$ under $P(z)$ also tend to ∞.

Claim 3: $[-2,2]$ is the Julia set for $P(z) = z^2 - 2$.

> This follows immediately from Theorem 3 since $[-2,2] = \partial\Omega$.

Now for the proof of Theorem 3 it is easy to see that $\partial A(\infty) \subset J(P)$. Suppose w_o is on the boundary of $\partial A(\infty)$ and V is any neighborhood of w_o. Then for $z \in A(\infty) \cap V$, $P^n(z) \to \infty$ but the iterates of w_o, $\{P^n(w_o)\}$ remain bounded. Thus $\{P^n\}$ is not normal in V so w_o is in the Julia Set. Details are in Carleson and Gamelin III.2.1.

To show $J(P) \subset \partial A(\infty)$ requires Montel's Criteria for Normality, a deep result of Complex Analysis. Theorem 6 is also a consequence of Montel's Criteria for Normality. The remainder of the proof of Theorem 3 and the proofs of Theorems 5, 6, and 7 are omitted in this presentation.

Theorem 5 ***Montel's Criteria for Normality*** *Let F be a family of meromorphic functions on a domain D. If there are three fixed values that are omitted by every $f \in F$, then F is a normal family. (Carleson and Gamelin I.3.1)*

Theorem 6 *The Julia set for R with deg$(R) \geq 2$ is perfect. (Carleson and Gamelin III.1.8)*

Theorem 7 *The Julia set for R with deg$(R) \geq 2$ is nonempty. (Carleson and Gamelin III.1.2)*

Theorems 8 and 9 are the main focus of this presentation and together completely characterize the Julia sets for quadratic polynomials. Also, any quadratic polynomial $P(z) = a_1z^2 + a_2z + a_3$ is conjugate to $P_c(z) = z^2 + c$ for some c in $\mathbb{C}_\infty$. First $P = g \circ S \circ g^{-1}, g = a_1z$ where $S(z) = z^2 + a_2z + a_1a_3$. Then $S = h \circ Q \circ h^{-1}, h = z + \frac{a_2}{2}$ gives $Q(z) = z^2 + c$ for $c = \frac{4a_1a_3+2a_2-a_2^2}{4}$.

Now, define z to be a **critical point** of a non-constant rational map R if R fails to be one-to-one in a neighborhood of z. Critical points occur where $R'(z) = 0$ or at poles of R with order two or higher. Define a **critical value** to be the image of a critical point. And, let a **branch** of the inverse map $P^{-1}(w)$ be the bijection between a neighborhood of w and a neighborhood of z where $P(z) = w$, w not a critical value of P.

Theorem 8 *The Julia set $J(P)$ where $P(z)$ is a polynomial with $deg(P) = 2$ is connected $\iff$ there is no finite critical point of P in the attracting basin of ∞, $A(\infty)$.*

First notice that ∞ is a superattracting fixed point of $P(z) = z^2 + c$. $P(\infty) = \infty$, and the derivative of P at ∞ is given by $\frac{1}{P'(\frac{1}{\zeta})} = \frac{\zeta}{2}$. Evaluating at $\zeta = 0$ gives $P'(\infty) = 0$.

Now in a neighborhood of ∞, there exists φ, a conformal map, such that

$$\varphi(P(z)) = \varphi(z)^2$$

where $\varphi(z) = z + O(1)$ at ∞ (this is the Boettcher Functional Equation for φ). That is the following diagram commutes:

$$\begin{tikzcd}
.\infty \arrow[r, "P(z)"] \arrow[d, "\varphi"'] & .\infty \arrow[d, "\varphi"'] \\
.\infty \arrow[r, "z^2"] & .\infty
\end{tikzcd}$$

Next since $\log|\varphi(z)|$ has a logarithmic pole at ∞, is positive and harmonic, and $\log|\varphi(z)| \to 0$ as $z \to \partial A(\infty)$, $\log|\varphi(z)| = G(z)$, Green's function for $A(\infty)$. Thus taking the logarithm of the modulus of the Boettcher functional equation yields a functional equation for $G(z)$,

$$G(P(z)) = 2G(z).$$

Now a component of the Fatou set maps onto another component of the Fatou set since otherwise a boundary point (an element of the Julia set) maps to a point in the interior of a component of the Fatou set. This is a contradiction since the Julia set is completely invariant. Next if a bounded component of $A(\infty)$ exists some iterate of P maps it onto the component of $A(\infty)$ which contains ∞. This means that for some z in the bounded component and integer n, $P^n(z) = \infty$. This is a contradiction because the iterates of a polynomial are polynomials and do not have poles. Thus $A(\infty)$ is connected.

Define a level curve of $G(z)$ as $\Lambda_a = \{z : G(z) = a\}$. Then $P(z)$ takes the level curve Λ_a to the level curve Λ_{2a} since for $z \in \Lambda_a$, $G(P(z)) = 2G(z) = 2a$. So $P(z) \in \Lambda_{2a}$. Define the exterior of a level curve Λ_a to be the set $E_a = \{z : G(z) > a\} = \{z : |\varphi(z)| > e^a\}$. Then $P(z)$ maps E_a two-to-one onto E_{2a} which is a subset of E_a.

To extend $\varphi(z)$ first consider a neighborhood $D = E_r$ of ∞ on which the Boettcher functional equation holds. Then on $E_{\frac{r}{2}}$ we can define $\varphi(z) = \sqrt{\varphi(P(z))}$ (since $z \in E_{\frac{r}{2}} \Rightarrow P(z) \in E_r$) so the right hand side of the equation is defined). Continue in this way defining $\varphi(z)$ on $E_{\frac{r}{2^n}} = \{z : |\varphi(z) > e^{\frac{r}{2^n}}\}$ as long as there are no critical points in the extended region. (At a critical point a single-valued analytic function cannot be defined). So as $n \to \infty$, φ is defined on $\bigcup_{n=1}^{\infty} E_{\frac{r}{2^n}} = \{z : |\varphi(z)| > 1\} = \{z : G(z) > 0\} = A(\infty)$.

($\Leftarrow$) Recall that $A(\infty)$ is connected. Now φ is a homeomorphism (continuous, 1-1, onto, and inverse is continuous) which maps $A(\infty)$ conformally to the exterior of the open unit disk. Since simple connectivity is preserved by a homeomorphism, and the exterior of the unit disk on the Riemann sphere is simply connected, $A(\infty)$ must be simply connected. Thus it follows that $\partial A(\infty)$ is connected.

($\Rightarrow$) Assume that $\exists z_o \in A(\infty)$, z_o a finite critical point of $P(z)$. Let $G(z_o) = r_o$ and consider Λ_{r_o}. Differentiating the functional equation at z_o yields $\left(\frac{\partial}{\partial z} G(P(z_o))\right) P'(z_o) = 2\frac{\partial}{\partial z} G(z_o)$. $P'(z_o) = 0$ since z_o is a critical point of $P(z)$ implies that $\frac{\partial}{\partial z} G(z_o) = 0$. So z_o is a critical point of $G(z)$. Thus the level curve Λ_{r_o} consists of at least two simple closed curves that meet at the critical point z_o.

Within each of these simple curves there exist points in the Julia set. If not, G(z) is harmonic and positive on a non-empty region U within one of the simple curves and the Maximum principle applied to $G(z)$ and

$-G(z)$ gives $G(z) \leq r_o$ and $-G(z) \leq -r_o$ for all $z \in U$. So $G(z) \equiv r_o$ on U. Let f be the analytic function with real part equal to $G(z)$. Then by the uniqueness theorem, $G(z) \equiv r_o$ on $A(\infty)$. This contradicts that $A(\infty) \to \infty$. Thus $J(P)$ is disconnected.

In fact, $J(P)$ has uncountably many components. Let w be an element of the backward orbit of z_o, that is $P^n(w) = z_o$ for some n. Then by repeatedly applying the functional equation, $G(P^n(w)) = 2^n G(w)$, or $G(w) = 2^{-n} G(P^n(w))$. Thus $G(w) = 2^n G(z_o)$. Differentiate both sides to get $\frac{\partial}{\partial z} G(w) = 2^n \frac{\partial}{\partial z} G(z_o) = 0 \Rightarrow w$ is a critical point of $G(z)$. Since the choice of w was arbitrary, every point in the backward orbit of a critical point is also a critical point. The level curves split at each of the w, so follow the splitting by assigning 0 to the left branch and 1 to the right branch. Since there are uncountably many sequences of 0's and 1's there are uncountably many components of J.

Theorem 9 *Let $P(z) = z^2 + c$. If $P^n(0) \to \infty$, then the Julia set $J(P)$ is totally disconnected.*

Since $\infty \in F(P)$ and $F(P)$ is open, there exists a neighborhood D_∞ of ∞ such that $\overline{D}_\infty \subset F(P)$. And since ∞ is an attracting fixed point of P, $P(\overline{D}_\infty) \subset D_\infty$. Let $D = \mathbb{C}_\infty - \overline{D}_\infty$. Then D is an open set and $J(P) \subset D$.

Now, since $P^n(0) \to \infty$ by assumption, choose N large so that P^N maps 0 to D_∞. Thus for $n \geq N$, there is no critical value of P^n in $\overline{D}$, and all branches of the inverse map P^{-n} are defined and map $\overline{D}$ into D. (Else $\exists z \in \overline{D}$ such that $w = P^{-n}(z) \in (\mathbb{C}_\infty - D) = D_\infty$. Now $P^n(w) = z \in \overline{D}$, but $P^n : \overline{D}_\infty \to D_\infty$ implies that $z \in D_\infty$. This contradicts the choice of $z \in \overline{D}$.)

Let $z_o \in J(P)$. Then $P^n(z_o) \in J(P)$ since the Julia set is completely invariant under P. Define f_n to be the branch of the inverse map P^{-n} which maps $P^n(z_o)$ to z_o. That is, $f_n(P^n(z_o)) = z_o$. Since f_n maps $\overline{D}$ into D, $\{f_n\}$ are uniformly bounded on $\overline{D}$. Note that by modifying the integer N above, $\{f_n\}$ are uniformly bounded on a neighborhood of $\overline{D}$. Thus $\{f_n\}$ is normal on $\overline{D}$.

Now $\forall z \in D \cap A(\infty)$, $f_n(z)$ accumulates on $J(P)$ since $f_n(z) \to \partial A(\infty)$ for all $z \in A(\infty)$ (except for $z = \infty$). Then let f be the limit of some subsequence $\{f^{n_k}\}$ of $\{f_n\}$. Now f maps $D \cap A(\infty)$ into J since

$f^n(z) \to w \in J(P)$ and $f^{n_k}(z) \to f(z)$ implies $f(z) = w \in J(P)$ by the uniqueness of limits. Then the Open Mapping Theorem gives f is constant since $J = \partial A(\infty)$ by Theorem 3 and the boundary of an open set has empty interior. (If z is in the interior of the boundary of an open set U then there exists a neighborhood of z contained entirely in ∂U. But for any $\varepsilon > 0, \exists z_1 \in D(z, \varepsilon) \cap U$, by the definition of the boundary set. This contradicts that U is open.)

Now, $diam\{f_n(D)\} \to 0$. (Suppose not. Then $\exists \varepsilon > 0$ and $\{f_{n_k}\}$ such that $diam\{f_{n_k}\} \geq \varepsilon$. $\{f_{n_k}\}$ is normal so $\exists\{f_{n_{k_j}}\}$, a subsequence, and f a limit function, such that $f_{n_{k_j}} \to f$ uniformly. By the argument in the previous paragraph, $f \equiv w_o$, a constant. Thus for a fixed branch, $f_{n_{k_j}}$, with $j \geq j_o, |f_{n_{k_j}}(z) - w_o| < \frac{\varepsilon}{3}$ for all $z \in \overline{D}$. Then $diam\{f_{n_{k_j}}(D)\} < \frac{2\varepsilon}{3}$, a contradiction.) Then since f_n is continuous, $f_n(\overline{D}) \subseteq \overline{f_n(D)}$, and $f_n(\overline{D})$ has diameter tending to zero.

Next, by the invariance of the Fatou set, $\partial D \subset F(P)$ implies that $f_n(\partial D) \subset F(P)$ and it is disjoint from $J(P)$. Now recall that for $z_o \in J(P)$, f_n was chosen so that $f_n(P^n(z_o)) = z_o$, and $f_n(P^n(z_o)) \subset f_n(D)$ since $P^n(z_o) \in J(P) \subset D$. Also, $f_n(D) \subset f_n(\overline{D})$, so $z_o \in f_n(\overline{D})$ for all n. Now $diam\{f_n(\overline{D})\} \to 0$ implies that $\{z_o\}$ must be a connected component of $J(P)$. To see this recall that $\overline{D}$ consists of elements of $J(P)$ and the boundary which is in $F(P)$. For any $\varepsilon > 0$, choose N such that $diam\{f_N(\overline{D})\} < \varepsilon$. Within this disc, the boundary is mapped to a curve that winds around elements of the Julia set in the interior. Thus only points in the Julia set that are within 2ε of each other will be elements of a connected component of $J(P)$. But since ε can be chosen arbitrarily small, eventually all points of the Julia set will be separated by the Fatou set, $\partial f_N(D)$ for large enough N. This implies that $J(P)$ is totally disconnected.

The final theorem provides a characterization of the Mandelbrot set and follows directly from Theorems 8 and 9. Let P_c be the quadratic polynomial $z^2 + c$. Then the Mandelbrot set consists of c such that $\{P_c^n(0)\}$ is bounded.

Theorem 10 *If $P_c^n(0) \to \infty$, then the Julia set $J(P_c)$ is totally disconnected. Otherwise, $\{P_c^n(0)\}$ is bounded, and the Julia set is connected.*

First notice that 0 is the only finite critical point for $P(z) = z^2 + c$ since $P'(z) = 2z$ so $P'(0) = 0$.

If $P^n(0)$ is bounded then $0 \notin A(\infty)$ and $J(P)$ is connected by Theorem 8.

Next, if $P^n(0) \to \infty$ then $J(P)$ is totally disconnected by Theorem 9.

The main results described in this presentation come from Carleson and Gamelin's text **Complex Dynamics**. Detailed proofs can be found in Chapters 3 and 8. The majority of the background material on iteration was covered in my reading course with Dr. Smith this semester (Chapters 1-4 of Alan F. Beardon's text **Iteration of Rational Functions**). Figures are from Peitgen and Richter's **The Beauty of Fractals**.