ON SEMIFREE SYMPLECTIC CIRCLE ACTIONS

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To Mom & Dad
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Abstract

In 1988, Dusa McDuff constructed the only known example of a non-Hamiltonian symplectic circle action with fixed points. On a Kähler manifold, any symplectic circle action with fixed points is Hamiltonian, so McDuff’s construction provides an example of a symplectic manifold which cannot support a Kähler structure. We prove some results giving cohomological constraints on generalizing McDuff’s construction, focusing on semifree symplectic circle actions on 6-manifolds with fixed surfaces.
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Preface

Consider the equatorial action of the circle group $S^1$ on the 2-sphere $S^2$. The orbits of the action are the circles of latitude and the fixed points are the poles. $S^2$ supports an oriented area measure called a symplectic structure which is invariant under the action. Assuming for simplicity that $S^1$ wraps the equator around itself just once provides an example of a semifree symplectic circle action. The study of such actions, especially on closed manifolds, illustrates the interplay between symplectic topology and equivariant cohomology. These actions have quite a different flavor than, on one hand, actions of compact semisimple groups and, on the other hand, toric varieties. The difference arises from the fact that most interesting symplectic actions, such as the one described above, are Hamiltonian, meaning that there are globally defined functions arising from conserved quantities à la Noether’s theorem. The existence of such functions, called Hamiltonians, allows one to bring in the powerful machinery of Morse theory, both ordinary and equivariant, to compute many of the topological invariants of the manifold. For circle actions, there arises the possibility that no such globally defined function exists.

In this dissertation, we highlight some facets of semifree symplectic circle actions which are amenable purely to the techniques of equivariant cohomology. We focus on the problem of finding extra conditions so that the existence of fixed points, a topological condition, implies the existence of the aforementioned globally defined Hamiltonian function. Just the existence of fixed points is not sufficient as McDuff has constructed an example of a non-Hamiltonian symplectic $S^1$-action with fixed points.

In Chapter 1, we review some basic material needed for the sequel. Chapter 2 reviews equivariant cohomology and equivariant characteristic classes. Section 2.6
provides a new construction of a useful normal equivariant characteristic class which greatly simplifies later calculations. Chapter 3 forms the heart of the dissertation, reviewing the necessary ingredients for the construction of non-Hamiltonian symplectic actions and providing in sections 3.5 and 3.6 some theorems concerning possible generalizations of the McDuff example. Section 3.7 provides a new proof of a Tolman-Weitsman result on semifree symplectic circle actions with isolated fixed points. Finally, in Chapter 4 we add some observations about other cohomological conditions vis-à-vis Hamiltonian versus non-Hamiltonian.
Chapter 1

Some Basic Material

1.1 Symplectic Vector Spaces

Let $W$ be an $2n$-dimensional real vector space.

**Definition 1.1.1.** A *symplectic structure* on $W$ is a skew-symmetric, nondegenerate bilinear form $\Omega$. The pair $(W, \Omega)$ is called a *symplectic vector space*.

Every symplectic vector space has a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ satisfying

\[
\Omega(e_i, e_j) = 0, \\
\Omega(f_i, f_j) = 0, \\
\Omega(e_i, f_j) = \delta_{ij},
\]

however, such a basis is far from unique. Not all linear subspaces of $W$ are symplectic since $\Omega$ must restrict to a nondegenerate form.

**Definition 1.1.2.** The *symplectic complement* of a linear subspace $V$ of $W$ is defined by

\[
V^\Omega = \{w \in W | \Omega(w, v) = 0 \text{ for all } v \in V\}.
\]
**Proposition 1.1.3.** For subspaces $U$ and $V$,

(i) $\dim V + \dim V^\Omega = \dim W$.

(ii) $(V^\Omega)^\Omega = V$.

(iii) $U \subseteq V \iff V^\Omega \subseteq U^\Omega$.

(iv) $V$ is symplectic $\iff V \cap V^\Omega = 0 \iff W = V \oplus V^\Omega$.

**Definition 1.1.4.** A subspace $V$ is *isotropic* if $V \subseteq V^\Omega$, *coisotropic* if $V^\Omega \subseteq V$, and *Lagrangian* if $V = V^\Omega$.

Every 1-dimensional subspace is isotropic and every codimension 1 subspace is coisotropic. An example of a Lagrangian subspace is that spanned by $\{e_1, \ldots, e_n\}$.

**Definition 1.1.5.** A *complex structure* $J$ on $W$ is a linear endomorphism satisfying $J^2 = -1$.

**Remark 1.1.6.** The 2-dimensional subspace spanned by $\{e_i, f_i\}$ is symplectic. Define $J$ on this subspace by $e_i \mapsto f_i$ and $f_i \mapsto -e_i$. Since $J^2 = -1$, it defines a complex structure. In fact, every symplectic vector space $(W, \Omega)$ has a complex structure $J$ which is said to be *compatible* with $\Omega$, meaning that $g(u, v) = \Omega(u, Jv)$ is a positive definite form. The construction of the operator $J$ can be made canonical if one first chooses a positive definite form $G$ on $W$. One gets $J = (AA^*)^{-\frac{1}{2}}A$ where $A$ is a skew-symmetric endomorphism of $W$ satisfying $\Omega(u, v) = G(Au, v)$. Also, $g(u, v) = G(AA^*u, v)$. See Cannas da Silva [Ca01] for details.
1.2 Symplectic Manifolds

**Definition 1.2.1.** Let $M^{2n}$ be a manifold. A smooth choice of symplectic structure on each tangent space defines a 2-form $\omega$ on $M$. If $\omega$ is closed, then the pair $(M, \omega)$ is said to be a *symplectic manifold*.

The nondegeneracy condition on each tangent space implies that $\omega^n$ is nonzero and thus defines a volume form on $M$, whence $M$ is orientable. For a closed manifold, $\omega$ is cannot be exact since otherwise $\omega^n$ would be exact and thus would have zero integral over $M$ by Stoke’s theorem.

The canonical example of a symplectic manifold is $\mathbb{R}^{2n}$ with symplectic form $\omega = \sum dx_idy_i$. The complex version is $\mathbb{C}^n$ with symplectic form $\omega = \frac{i}{2} \sum dz_k d\bar{z}_k$ where $z_k = x_k + iy_k$. It turns out that locally, every symplectic manifold looks like the preceding example.

**Theorem 1.2.2** (Darboux). *Let $M^{2n}$ be a symplectic manifold and $p \in M$. Then there is a chart $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at $p$ such that $\omega = \sum dx_idy_i$ on $U$.*

**Definition 1.2.3.** A *symplectomorphism* between two symplectic manifolds $(M_1, \omega_1)$ and $(M_2, \omega_2)$ is a diffeomorphism $\phi : M_1 \to M_2$ such that $\phi^*(\omega_2) = \omega_1$.

Darboux’s theorem implies that there are no local symplectic invariants other than dimension. It is a special case of the much more general Weinstein theorem.

**Theorem 1.2.4** (Weinstein [We71]). *Let $\omega_0$ and $\omega_1$ be symplectic forms on a manifold $M$ agreeing on a closed submanifold $N$. Then there are neighborhoods $U_0$ and $U_1$ of $N$ and a symplectomorphism $\phi : U_0 \to U_1$ satisfying $\phi^*(\omega_1) = \omega_0$ and $\phi|_N = 1_N$.***
The proof uses what is known as the Moser trick which was originally used to prove

**Theorem 1.2.5** (Moser [Mo65]). Let \( \omega_0 \) and \( \omega_1 \) be symplectic forms on a compact manifold \( M \). Suppose that for \( \omega_t = (1 - t)\omega_0 + t\omega_1 \), \( 0 \leq t \leq 1 \), \( [\omega_t] \in H^2(M) \) is independent of \( t \), or equivalently, \( [\omega_1] = [\omega_0] \). Then there is an isotopy \( \rho_t : M \rightarrow M \) with \( \rho^*_t(\omega_t) = \omega_0 \) for all \( 0 \leq t \leq 1 \).

**Definition 1.2.6.** Symplectic forms \( \omega_0 \) and \( \omega_1 \) on \( M \) are deformation equivalent if there is a smooth family of symplectic forms \( \omega_t \), \( 0 \leq t \leq 1 \). They are isotopic if \( [\omega_t] \) is independent of \( t \), and they are strongly isotopic if there is an isotopy \( \rho_t : M \rightarrow M \) with \( \rho^*_1(\omega_1) = \omega_0 \).

Thus,

\[
\text{Strongly Isotopic } \Rightarrow \text{ Isotopic } \Rightarrow \text{ Deformation Equivalent.}
\]

Moser’s theorem can be used to prove that

\[
\text{Isotopic } \Rightarrow \text{ Strongly Isotopic.}
\]

**Remark 1.2.7.** Since locally, any closed form is exact, locally, \( \omega = -d\theta \), and \( \theta \) is called a symplectic potential. In Darboux coordinates, \( \theta = \sum y_i dx_i \). The sign convention comes from the construction of symplectic forms on cotangent bundles which is motivated by physics. For 1-dimensional mechanics, the phase space is \( T^*\mathbb{R} \cong \mathbb{R}^2 \) with coordinates \( (q,p) = (\text{position, momentum}) \) and symplectic form \( dqdp \). In this case, there is a global symplectic potential called the tautological 1-form \( \theta = pdq \).
1.3 \( S^1 \)-Actions

Given an \( S^1 \)-action on \( M \), there is an infinitesimal action of the Lie algebra \( \mathfrak{h} \) of \( S^1 \) on \( C^\infty(M) \) defined for \( X \in \mathfrak{h} \) by

\[
X(f)(x) = \left. \frac{d}{dt} \right|_{t=0} f(\exp(-tX) \cdot x).
\]

Similarly, one can define an operator \( L_X \) on the sections \( s \) of any tensor bundle over \( M \) by

\[
L_X(s) = \left. \frac{d}{dt} \right|_{t=0} \exp(-tX) \cdot s.
\]

On forms, \( L_X \) is a derivation of degree 0. Along with the exterior derivative \( d \) and interior multiplication \( i_X \), one has the Cartan homotopy formula

\[
L_X = di_X + i_Xd. \tag{1.3.1}
\]

**Definition 1.3.1.** For a symplectic action, \( L_X\omega = 0 \), so from the above formula, \( i_X\omega \) is a closed 1-form. If \( i_X\omega \) is exact, then the action is called *Hamiltonian*.

**Definition 1.3.2.** An action is *almost free* if every isotropy subgroup is finite. Every \( S^1 \)-action is almost free on the complement of its fixed point set. An action is *semifree* if every isotropy group is either trivial or the whole group.

**Example 1.3.3.** Take \( S^2 \) with symplectic form \( d\phi d\cos \theta \). The \( S^1 \)-action is given by rotation about the polar axis where north pole is defined by \( \cos \theta = 1 \) and the south pole is defined by \( \cos \theta = -1 \). We are abusing notation here as \( S^2 \) cannot be parametrized by one chart.

**Example 1.3.4.** Take \( T^2 = S^1 \times S^1 \) with symplectic form \( d\phi d\theta \). The \( S^1 \)-action is given by rotation on the first factor.
The first example is a Hamiltonian action and has fixed points. It would be nice to specify to what extent Hamiltonian actions are characterized by the existence of fixed points. The following are some results in this direction:

- A symplectic $S^1$-action on a Kähler manifold is Hamiltonian iff it has a fixed point. [Fr59]
- A symplectic $S^1$-action on a manifold whose cohomology algebra satisfies the weak Lefschetz condition, namely, $\wedge \omega^{n-1} : H^1 \cong H^{2n-1}$, is Hamiltonian iff it has a fixed point. [On88]
- A symplectic $S^1$-action on a symplectic 4-manifold is Hamiltonian iff it has a fixed point. [Mc88]
- A semifree symplectic $S^1$-action with isolated fixed points is Hamiltonian. [TW00]

Since every closed form is locally exact, every symplectic action is locally Hamiltonian. However, McDuff [Mc88] has given an example of a semifree symplectic $S^1$-action on a 6-manifold with fixed point set consisting of 2-tori which is not globally Hamiltonian.

### 1.4 Hamiltonians and Poisson Brackets

**Definition 1.4.1.** Given a vector field $X$ on $M$, if $i_X \omega$ is exact, then $i_X \omega = dh$ for some smooth function $h$ on $M$. Such a function is called a *Hamiltonian* of $X$. It is uniquely defined up to the addition of a locally constant function. If $X$ is the infinitesimal generator of a Hamiltonian $S^1$-action, then $h$ is called a *moment map*. By the nondegeneracy of $\omega$, given any smooth function $f$, there is a vector field $X_f$ such that $i_{X_f} \omega = df$. $X_f$ is called the *symplectic gradient* of $f$. 
In the above example of the $S^1$-action on $S^2$, the infinitesimal action is given by the vector field $X_\phi$. Since $i_{X_\phi}\omega = d \cos \theta$, the Hamiltonian is $z = \cos \theta$. For the $S^1$-action on $T^2$, $i_{X_\phi}\omega = d \theta$ and even though locally there is a well-defined angle function $\theta$ on $T^2$, globally, one does not exist.

**Definition 1.4.2.** An almost complex structure $J$ on a manifold $M^{2n}$ is a smooth choice of complex structure on each tangent space. This is equivalent to a reduction of the structure group of $TM$ from $O(2n)$ to $U(n)$.

**Remark 1.4.3.** Given a Riemannian metric on a symplectic manifold $(M, \omega)$, one can construct an almost complex structure $J$ compatible with $\omega$, meaning that $g(u, v) = \omega(u, Jv)$ is a Riemannian metric. This construction is canonical since it is canonical pointwise by Remark 1.1.6 and since there is a smooth square root on the bundle $\text{Hom}(TM, TM)$. $(\omega, g, J)$ is called a compatible triple. One can use the metric $g$ to define the metric gradient operator $\text{grad}$. There is a relationship between the metric gradient and the symplectic gradient

$$JX_f = \text{grad} f$$

which, in particular, implies that $g(X_f, \text{grad} f) = 0$.

Consider $\mathbb{R}^2$ with $\omega = dq dp$. Let $H = \frac{p^2}{2m} + V(q)$. This is the standard setup for motion in one dimension with potential $V$. Hamilton’s equations give the time evolution of $q$, position, and $p$, momentum,

$$\frac{dq}{dt} = \frac{\partial H}{\partial p},$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}.$$
The symplectic and metric gradients are, respectively,

\[ X_H = \frac{\partial H}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial}{\partial p} \]

and

\[ \text{grad} H = \frac{\partial H}{\partial q} \frac{\partial}{\partial q} + \frac{\partial H}{\partial p} \frac{\partial}{\partial p} \]

One can verify that \( JX_H = \text{grad} H \) where \( J \) is defined by \( J(\frac{\partial}{\partial q}) = \frac{\partial}{\partial p} \) and \( J(\frac{\partial}{\partial p}) = -\frac{\partial}{\partial q} \).

For \( S^1 \), the canonical example of a Hamiltonian action arises from multiplication in \( \mathbb{C} \). Recall that the symplectic structure on \( \mathbb{C} \) is given by \( i z \overline{d}z \overline{d} \overline{z} \). For the action defined by \( z \mapsto e^{i\lambda \theta} \cdot z \), the infinitesimal action and moment map are

\[ X_h = i\lambda \left( z \frac{\partial}{\partial z} - \overline{z} \frac{\partial}{\partial \overline{z}} \right) \quad (1.4.1) \]

and

\[ h = -\frac{1}{2} \lambda |z|^2 \quad (1.4.2) \]

where \( \lambda \) is an integer called the weight of the action. Through the equations

\[
\begin{align*}
\frac{\partial}{\partial z} &= \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \\
\frac{\partial}{\partial \overline{z}} &= \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),
\end{align*}
\]

one gets

\[ X_h = \lambda \left( -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) \]

which is the infinitesimal action of a speed \( \lambda \) rotation about the origin in \( \mathbb{R}^2 \). By speed \( \lambda \), it is meant that the circle wraps around the origin \( \lambda \) times, as is clear from the action given above.
The algebra of smooth functions $C^\infty(M)$ on a symplectic manifold $M$ has a Lie algebra structure given by the Poisson bracket which is defined by

$$\{f, g\} = \omega(X_f, X_g)$$

$$= i_{X_f} \omega(X_g)$$

$$= df(X_g)$$

$$= X_g f.$$

On $(\mathbb{R}^2, dqdp)$, the Poisson bracket has the form

$$\{f, g\} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q}.$$

Hamilton’s equations can be recast as

$$\frac{dq}{dt} = \{q, H\},$$

$$\frac{dp}{dt} = \{p, H\}.$$

Remark 1.4.4. This formulation of mechanics is crucial to understanding the relationship between the classical and quantum theories. For example, Heisenberg’s uncertainty principle is derived from the fact that $[Q, P] \neq 0$ for the quantum position, $Q$, and momentum, $P$, observables. Classically, this is detected by $\{q, p\} \neq 0$.

### 1.5 The Equivariant Darboux Theorem

In [Fr59], Frankel shows that the fixed point components of a symplectic $S^1$-action are symplectic. Given a symplectic $S^1$-manifold $M$, for $p \in F$ a component of the fixed point set, it is clear that $T_p F$ and $N_p F$, the normal space of $F$ at $p$, are symplectic
subspaces of $T_pM$ and $T_pM = T_pF \oplus N_pF$. Also, the linear action of $S^1$ on $N_pF$ is symplectic. There is an extension of this to charts by an equivariant version of the Darboux theorem for the neighborhood of a fixed point due to Weinstein [We77].

**Theorem 1.5.1** (Equivariant Darboux). Let $(M^{2n}, \omega)$ be a symplectic $S^1$-manifold with $p \in F^{2k}$ a component of the fixed point set. Then there is an $S^1$-equivariant symplectic chart $\phi : (\mathbb{R}^{2n}, \sum dx_i dy_i) \to (M, \omega)$ centered at $p$ where $\mathbb{R}^{2n}$ has a symplectic representation of $S^1$. The subspace $V^{2k} \subseteq \mathbb{R}^{2n}$ where the representation is trivial provides Darboux coordinates on $F$ under $\phi$.

Since the symplectic representation of $S^1$ on $N_pF$ must preserve a compatible almost complex structure on $N_pF$, the representation must also be complex and thus must split into a sum of 1-dimensional complex representations, $N_pF = \bigoplus_{i=1}^k E_i$. Each representation $E_i$ has the form given by (1.4.1). The associated local moment map (1.4.2) yields a local interpretation of the fixed point set as the locus of attraction or repulsion of an harmonic oscillator potential depending on whether $\lambda$ is negative or positive. Such a splitting does not exist at the bundle level, but we do have a splitting $NF = \bigoplus \lambda E_{\lambda}$ over the eigenvalues $\lambda$ of the representation. In the case of a semifree action, we have $NF = E_+ \oplus E_-$ where $E_+$ is the $(+1)$-eigenbundle and $E_-$ is the $(-1)$-eigenbundle.

### 1.6 Morse Theory

**Definition 1.6.1.** A smooth function $f : M \to \mathbb{R}$ is a *Morse function* if the critical points $p$ of $f$ are nondegenerate, meaning that the Hessian matrix $H_p(f) = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)$ is nonsingular.

Nondegeneracy does not depend on the choice of coordinates and, furthermore, implies that the critical points are isolated.
Definition 1.6.2. The index $\lambda_p(f)$ of a critical point $p$ is the number of negative eigenvalues of $H_p(f)$. It also does not depend on the choice of coordinates.

The Betti numbers of $M$, $b_i(M)$, are constrained by the numbers of critical points of each index, $c_i(f)$, by the Morse inequalities (1.6.1). Let

$$M_t(f) = \sum_p t^{\lambda_p(f)} = \sum_i c_i(f) \cdot t^i$$

be the Morse counting series and

$$P_t(M) = \sum b_i(M) \cdot t^i$$

be the Poincaré series. Then there is a polynomial $R(t)$ with nonnegative coefficients satisfying

$$M_t(f) - P_t(M) = (1 + t) \cdot R(t). \quad (1.6.1)$$

Definition 1.6.3. A Morse function is perfect if $R(t) = 0$, whence $b_i(M) = c_i(f)$.

A CW-approximation to $M$ can be built inductively from a Morse function by attaching a $k$-disk for each critical point of index $k$.

Bott [Bo54] extended Morse theory to include functions, called Morse-Bott functions, whose critical point sets consist of submanifolds.

Definition 1.6.4. A nondegenerate critical submanifold is a submanifold $C$ on which $df = 0$ and $H_p(f)$ is nonsingular when restricted to the normal space $N_pC$.

The index $\lambda_C(f)$ of a nondegenerate critical submanifold $C$ is the number of negative eigenvalues of this restriction. From the splitting $NC = N^+_C \oplus N^-_C$ of the
normal bundle into positive and negative eigenbundles of $H_C(f)$, the index is well-defined. This is a generalization of the isolated critical point case. The *Morse-Bott inequalities* are now given by

$$M_{t,C}(f) - P_t(M) = (1 + t) \cdot R(t)$$

where $R(t)$ is again a polynomial with nonnegative coefficients and

$$M_{t,C}(f) = \sum_C P_t(C) \cdot t^{\lambda_C(f)}$$

is the *Morse-Bott counting series*.

We again have an inductive construction of a CW-approximation from the critical point set, but we now attach the disk bundle of the negative normal bundle $N_-C$ for each critical submanifold $C$.

The upshot of this digression into Morse-Bott theory is that a Hamiltonian $S^1$-manifold supports a Morse-Bott function, the Hamiltonian, whose critical point set is exactly the fixed point set of the action. Moreover, as the fixed point set consists of symplectic submanifolds, the Morse-Bott indices are all even, whence the Hamiltonian is perfect.
Chapter 2

Equivariant Characteristic Classes

2.1 The Borel Construction

Let $G$ be a compact, connected Lie group. Given a principal $G$-bundle $P \to B$ and a $G$-space $F$, one can form the associated bundle

$$
\begin{array}{c}
F \\
\downarrow
\end{array} \quad \begin{array}{c}
P \times_G F \\
\downarrow \\
B
\end{array}
$$

where the free action of $G$ on $P \times F$ is given by $g \cdot (p, f) = (p \cdot g^{-1}, g \cdot f)$. Since a $G$-bundle is completely determined by its transition functions, any $G$-bundle can be realized as an associated bundle of some principal $G$-bundle. By a construction of Milnor [Mi56], there is a universal principal $G$-bundle $EG \to BG$ where $EG$ is contractible. Any principal $G$-bundle $P \to B$ can be realized as the pullback bundle of some map $f : B \to BG$. Maps homotopic to $f$ induce isomorphic bundles. $BG$ is called the classifying space of $G$. 

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Definition 2.1.1. Given a $G$-manifold $M$, the associated bundle to the universal $G$-bundle
\[ M \longrightarrow EG \times_G M \]
\[ \downarrow \]
\[ BG \]
is called the Borel construction.

Definition 2.1.2. The ordinary cohomology of $M_G = EG \times_G M$ is called the equivariant cohomology of $M$,
\[ H_G^*(M) = H^*(M_G). \]

Remark 2.1.3. We take all cohomology with coefficients in $\mathbb{R}$.

If the action is free, then $H_G^*(M) = H^*(M/G)$. On the other hand,
\[ H_G^*(pt) = H^*(BG). \]

The Serre spectral sequence for the fibration $M_G \to BG$ is a guide to compute $H_G^*(M)$, though in practice such a computation may be extremely difficult, if not intractable. Recall that $E_2^{p,q} = H^p(BG) \otimes H^q(M)$. The equivariant cohomology functor $H_G^*$ satisfies all but the dimension axiom of the Eilenberg-Steenrod axioms, so there is a Mayer-Vietoris sequence for invariant sets which facilitates an alternate method of calculation of $H_G^*(M)$.

It is useful for certain constructions to work with a finite dimensional smooth $N$-approximation
\[ M \longrightarrow M_{G,N} = EG_N \times_G M \]
\[ \downarrow \]
\[ BG_N \]
to the Borel construction where $EG_N \to BG_N$ is an $N$-universal $G$-bundle, namely,
a principal $G$-bundle with $(N - 1)$-connected total space. $H^*_{G,N}(M) = H^*(M_{G,N})$ is an approximation to $H^*_G(M)$ and we have

$$
EG = \lim \rightarrow EG_N,
$$

$$
BG = \lim \rightarrow BG_N,
$$

$$
M_G = \lim \rightarrow M_{G,N},
$$

$$
H^*_G(M) = \lim \leftarrow H^*_{G,N}(M).
$$

\hspace{1cm} (2.1.1)

2.2 The Weil Model

There is an equivariant form of the de Rham theorem. The \textit{Weil model} gives an appropriate \textit{differential graded algebra}, \textit{DGA}, analogous to the de Rham algebra of forms. Denote the Lie algebra of $S^1$ by $\mathfrak{h}$. The \textit{Weil algebra}

$$
W_{\mathfrak{h}} = \Lambda \mathfrak{h}^* \otimes S\mathfrak{h}^* = \mathbb{R}[\theta, u]/(\theta^2)
$$

has generator $\theta$ of $\Lambda \mathfrak{h}^*$ of degree 1 and generator $u$ of $S\mathfrak{h}^*$ of degree 2. The differential $D$ on $W_{\mathfrak{h}}$ is defined by $D\theta = u$ and $Du = 0$. As $\{1, u, u^2, \ldots, \theta u, \theta u^2, \ldots\}$ is a basis for $W_{\mathfrak{h}}$ and $u^k = D(\theta u^{k-1})$, $W_{\mathfrak{h}}$ is acyclic. One should think of $W_{\mathfrak{h}}$ as an algebraic model of $ES^1 = S^\infty$; $\theta$ corresponds to the connection 1-form for the bundle $ES^1 \rightarrow BS^1$ and $u$ corresponds to the curvature 2-form which represents the pullback of the universal first Chern class in $H^2(BS^1)$. 
There is an action of $\mathfrak{h}$ on $W\mathfrak{h}$ given by

\begin{align*}
i_X \theta &= 1, \\
i_X u &= 0, \\
L_X &= i_X D + Di_X,
\end{align*}

where $X$ is a generator of $\mathfrak{h}$. Elements in a DGA that satisfy $i_X a = L_X a = 0$ are called basic. Consider a principal $S^1$-bundle $P \to B$. Any form $\eta \in \Omega^* P$ which is invariant, $L_X \eta = 0$, and horizontal, $i_X \eta = 0$, must be a pullback of some form $\tilde{\eta} \in \Omega^* B$, hence the term. Thus, $(W\mathfrak{h})_{bas} = S\mathfrak{h}^*.$

The Weil model $\Omega^*_\mathfrak{h}(M)$ is the DGA of basic elements of $\Omega^* M \otimes W\mathfrak{h}$. The differential is given by $D = d \otimes 1 + 1 \otimes D$. The action of $\mathfrak{h}$ is defined by $I_X = i_X \otimes 1 + 1 \otimes i_X$ and $L_X = L_X \otimes 1 + 1 \otimes L_X$. Following the treatment of Atiyah-Bott [AB84], any $\eta \in \Omega^* M \otimes W\mathfrak{h}$ can be written as

$$\eta = \sum_j \alpha_j \otimes u^j + \sum_k \beta_k \otimes \theta u^k$$

where $j, k \in \mathbb{N}$ and $\alpha_j, \beta_k \in \Omega^* M$. So $\eta$ is basic if

$$I_X \eta = \sum_j i_X \alpha_j \otimes u^j + \sum_k (i_X \beta_k \otimes \theta u^k + (-1)^{\deg \beta_k} \beta_k \otimes u^k) = 0$$

and

$$L_X \eta = \sum_j L_X \alpha_j \otimes u^j + \sum_k L_X \beta_k \otimes \theta u^k = 0$$

which gives

$$i_X \alpha_j + (-1)^{\deg \beta_j} \beta_j = 0$$
and

\[ L_X \alpha_j = 0. \]

Remark 2.2.1. The Weil model should be thought of as an equivariant de Rham model of the Borel construction. See Guillemin-Sternberg [GS99].

### 2.3 The Cartan Model

Consider the endomorphism \( \gamma = i_X \otimes \theta \) of \( \Omega^*_h(M) \). Since \( \gamma^2 = 0 \) we have automorphisms \( e^\gamma = 1 + \gamma \) and \( e^{-\gamma} = 1 - \gamma \). Since for \( \eta \in \Omega^*_h(M) \)

\[ \gamma(\eta) = \sum_j (-1)^{\deg \alpha_j} i_X \alpha_j \otimes \theta u^j, \]

we get

\[ e^{-\gamma}(\eta) = \sum_j \alpha_j \otimes u^j, \]

and so

\[ \text{im}(e^{-\gamma}) = \Omega^*_\text{inv} M \otimes S\mathfrak{h}^* \]

\[ \simeq \Omega^*_\text{inv} M[u]. \]

Thus, \( e^{-\gamma} \) yields an isomorphism \( \Omega^*_h(M) \simeq \Omega^*_\text{inv} M[u] \) called the Mathai-Quillen isomorphism [MQ86]. \( \Omega^*_\text{inv} M[u] \) is called the Cartan model and has differential defined by \( d_X = e^{-\gamma} D e^\gamma \). On a basis \( \{ \alpha \otimes 1, 1 \otimes u \} \) we get

\[ d_X \alpha = d\alpha - i_X \alpha \cdot u, \]

\[ d_X u = 0. \quad (2.3.1) \]
By (2.3.1), an element $\zeta = \sum_j \alpha_j \otimes u^j \in \Omega^*_{inv} M[u]$ is equivariantly closed, $d_X(\zeta) = 0$, iff

$$d\alpha_{j-1} = i_X \alpha_j \text{ for } i \geq 1.$$

(2.3.2)

**Remark 2.3.1.** If $(M, \omega)$ is a symplectic $S^1$-manifold, then the action is Hamiltonian iff $\omega$ has an equivariantly closed extension $\omega^# = \omega + \mu \cdot u$ in the Cartan model where $\mu : M \to \mathfrak{h}^*$ is the equivariant moment map. In the Hamiltonian case, we would also obtain an extension of $\omega$ in the Weil model

$$\bar{\omega} = e^\gamma(\omega^#)$$

$$= \omega \otimes 1 + \mu \otimes u + d\mu \otimes \theta$$

(2.3.3)

$$= \omega \otimes 1 + D(\mu \otimes \theta).$$

### 2.4 Equivariant Localization

Let $M$ be a closed $G$-manifold. The equivariant cohomology $H^*_G(M)$ is an $H^*(BG)$-module under the pullback $\pi_G^*$ of $M \xrightarrow{\pi} pt$. The *equivariant Gysin homomorphism*, or *equivariant pushforward*, of $\pi$

$$\pi_G^* : H^*_G(M) \to H^*(BG)$$

can be interpreted as *integration over the fibre*

$$\pi_G^*(\eta) = \int_M \eta$$
in the Borel construction. Let $F$ be the fixed point set of the action. The equivariant pushforward $i^G_i$ of $F \rightarrow M$ satisfies

$$i^*_G i^G_i(1) = e(\nu_F)$$

where $e(\nu_F)$ is the equivariant Euler class of the normal bundle $\nu_F$ of $F$ in $M$. If $e(\nu_F)$ is invertible in some localization $S^{-1}H^*_G(F)$, then

$$\frac{i^*_G}{e(\nu_F)}$$

will be an inverse to $S^{-1}H^*_G(F) \xrightarrow{i^G} S^{-1}H^*_G(M)$. This will indeed be so if $S$ is generated by the restrictions of $e(\nu_F)$ to a point $p_i$ in each component $F_i$ of $F$. This is the content of the localization theorem. The diagram

$$S^{-1}H^*_G(F) \xrightarrow{i^G} S^{-1}H^*_G(M)$$

provides the integration formula

$$\int_M \eta|_M = \int_F \frac{\eta|_F}{e(\nu)} = \sum_i \int_{F_i} \frac{\eta|_{F_i}}{e(\nu_i)}$$

(2.4.1)

where $\eta \in H^*_G(M)$ and the $F_i$ are the components of $F$. The above treatment is by Atiyah-Bott [AB84].

Let $M$ be a closed $G$-manifold. The equivariant cohomology $H^*_G(M)$ is an $H^*(BG)$-module. Let $F$ be the fixed point set of the action. We have the following exact
sequence of $H^*(BG)$-modules

$$H^*_G(M, M - F) \to H^*_G(M) \to H^*_G(M - F)$$

Let $i^*_G,k$ be the pullback of the inclusion of a point $p_k$ in a component $F_k$ of $F$. If we localize at the multiplicative set $\mathcal{S}$ generated by the pullbacks $i^*_G,k(\eta_k)$ of the equivariant Euler class $\eta_k$ of $F_k$, we end up with an isomorphism of $\mathcal{S}^{-1}H^*(BG)$-modules

$$\mathcal{S}^{-1}H^*_G(M, M - F) \to \mathcal{S}^{-1}H^*_G(M)$$

since $H^*_G(M - F)$ is $\mathcal{S}$-torsional. We can choose the support of $\eta$ close to $F$, so the restriction of $\eta$ to a suitable open subset of $M - F$ is zero.

By excision, we obtain the exact sequence of $H^*(BG)$-modules

$$H^*_G(DF, SF) \to H^*_G(DF) \to H^*_G(SF)$$

where $SF$ and $DF$ are the normal sphere and disk bundles of $F$, respectively. Using excision again to shrink the support of $\eta$ and then localizing at $\mathcal{S}$ yields an isomorphism of $\mathcal{S}^{-1}H^*(BG)$-modules

$$\mathcal{S}^{-1}H^*_G(DF, SF) \to \mathcal{S}^{-1}H^*_G(DF).$$

Since $F$ is a deformation retract of $DF$, we have the localization isomorphism

$$\mathcal{S}^{-1}H^*_G(M) \to \mathcal{S}^{-1}H^*_G(F).$$

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2.5 Equivariant Characteristic Classes

Let $G$ and $S$ be compact, connected Lie groups and $M$ and $P$ be $S$-manifolds.

**Definition 2.5.1.** A principal $G$-bundle $P \xrightarrow{\pi} M$ is $S$-equivariant if $s(gp) = g(sp)$ and $\pi(sp) = s\pi(p)$ for all $g \in G$, $s \in S$ and $p \in P$.

We get a principal $G$-bundle $P_S \xrightarrow{\pi_S} M_S$ of Borel constructions which is functorial. Let $ES \xrightarrow{\rho} BS$ be the universal principal $S$-bundle and let $ES_N \xrightarrow{\rho_N} BS_N$ be a finite dimensional smooth approximation. The diagrams of bundles

\[
\begin{array}{ccc}
P & \longrightarrow & P_S \\
\pi & \downarrow & \pi_S \\
M & \longrightarrow & M_S
\end{array}
\]

and

\[
\begin{array}{ccc}
ES \times P & \longrightarrow & P_S \\
\downarrow & & \downarrow \\
ES \times M & \longrightarrow & M_S
\end{array}
\]

have approximations

\[
\begin{array}{ccc}
P & \longrightarrow & P_{S,N} = ES_N \times_S P \\
\pi & \downarrow & \pi_{S,N} \\
M & \longrightarrow & M_{S,N} = ES_N \times_S M
\end{array}
\]

and

\[
\begin{array}{ccc}
ES_N \times P & \longrightarrow & P_{S,N} \\
\downarrow & & \downarrow \\
ES_N \times M & \longrightarrow & M_{S,N}
\end{array}
\]

can be used to show that the equivariant characteristic classes of the bundle $\pi_S$ are inverse limit classes of ordinary characteristic classes of the bundles $\pi_{S,N}$.
Suppose that \( M^{2n} \) is almost complex, that \( G = U(n) \), and that \( S \) preserves the almost complex structure. If the \( S \)-action on \( M \) is trivial, then \( S \)-equivariance implies that \( S \) commutes with \( G \) in each fibre of \( \pi \). Furthermore, \( M_S = BS \times M \) and so \( H^*_S(M) = H^*(BS) \otimes H^*(M) \). Thus, in this case, the total Chern class \( c(\pi_S) \) must satisfy
\[
c(\pi_S) = c(\pi) \otimes c(\rho).
\]

The total Chern class \( c(M_{S,N}) \in H^*(M_{S,N}) = H^*_{S,N}(M) \) restricts to \( c(M) \in H^*(M) \) under \( M \to M_{S,N} \) and for the fixed point set \( F \) of the \( S \)-action on \( M \) restricts to
\[
c(F_{S,N}) \cdot c(\nu_{F_{S,N}}) \in H^*(F_{S,N}) = H^*_{S,N}(F)
\]
where \( \nu_{F_{S,N}} \) is the normal bundle of \( F_{S,N} \) in \( M_{S,N} \).

### 2.6 A Normal Characteristic Class

Let \( G \) be a compact, connected Lie group. Suppose \( M \) is a closed almost complex \( G \)-manifold with an invariant almost complex structure. Let
\[
\begin{array}{ccc}
M & \xrightarrow{i_N} & M_{G,N} \\
\downarrow{\pi_N} & & \downarrow{\pi_N} \\
BG_N & & 
\end{array}
\]

be an \( N \)-approximation to the Borel construction. The almost complex structures on \( T_sM \) and \( T_sBG_N \) together with \( G \)-invariance yields an almost complex structure on \( T_sM_{G,N} \), so we have Chern classes
\[
c_i(M_{G,N}) \in H^*_{G,N}(M).
\]
**Definition 2.6.1.** Let

\[ c_i(M_G) \in H_G^*(M) \]

be the inverse limit class of \( c_i(M_{G,N}) \) given by (2.1.1). \( c_i(M_G) \) is called the \( i^{th} \) equivariant Chern class of \( M \).

The total Chern class \( c(M_{G,N}) \) restricts to \( c(M) \) under the inclusion of the fibre,

\[
i_N^*(c(M_{G,N})) = c(M) \cdot c(\nu_M)
\]

\[
= c(M)
\]

(2.6.1)

where \( \nu_M \) is the normal bundle of \( M \) in \( M_{G,N} \), since \( c(\nu_M) = 1 \) by local triviality of \( \pi_N \). Since

\[
i_N^* \pi_N^*(c(BG_N)) = 1,
\]

(2.6.2)

\( \pi_N^*(c(BG_N)) \) is invertible in \( H_{G,N}^*(M) \), so the class

\[
\eta_N = \frac{c(M_{G,N})}{\pi_N^*(c(BG_N))} \in H_{G,N}^*(M)
\]

also restricts to \( c(M) \) under the inclusion of the fibre. Restricting \( \eta_N \) to the fixed point set \( F \) under the inclusion \( i_F : F_{G,N} \to M_{G,N} \), we obtain

\[
i_{F,N}^*(\eta_N) = c(F) \cdot c(\nu_{F_{G,N}}) \in H_{G,N}^*(F)
\]

(2.6.3)

where \( \nu_{F_{G,N}} \) is the normal bundle of \( F_{G,N} \) in \( M_{G,N} \). Since equations (2.6.1), (2.6.2) and (2.6.3) are independent of \( N \) for large \( N \), we have

**Theorem 2.6.2.** Let \( M \) be a closed almost complex manifold, \( G \) be a compact, connected Lie group preserving the almost complex structure, and \( F \) be the fixed point set
of the action. The class $\pi^*(c(BG))$ is invertible in $H_G^*(M)$ so we may define the class

$$\eta = \frac{c(M_G)}{\pi^*(c(BG))} \in H_G^*(M)$$

which satisfies

$$i^*(\eta) = c(M)$$

and which under the inclusion $i_F: F \to M$ satisfies

$$i_F^*(\eta) = c(F) \cdot c(M_G) \in H_G^*(F).$$

In the sequel, we will denote $\frac{c(M_G)}{\pi^*(c(BG))}$ by $c(M_G)/c(BG)$.

Remark 2.6.3. Let $(\omega, g, J)$ be a compatible triple with $\omega$ and $g$ invariant. Then $J$ is invariant. Thus, Theorem 2.6.2 applies to symplectic $G$-actions.
Chapter 3

McDuff-Type Constructions

3.1 Symplectic Reduction

Let \((M, \omega)\) be a Hamiltonian \(S^1\)-manifold with moment map \(\mu\). Assume that the action is semifree and that 0 is a regular value of \(\mu\). The level set \(\mu^{-1}(0)\) is a codimension 1 submanifold of \(M\) and thus is coisotropic. See Definition 1.1.4. \(\Omega\) decends to a symplectic structure on \(W/W^\Omega\) and

\[
\dim W/W^\Omega = \dim W - \codim W.
\]

It turns out that the symplectic complement of the tangent space of \(\mu^{-1}(0)\) is spanned by \(X_{S^1}\), the fundamental vector field of the action, so the orbit space \(\mu^{-1}(0)/S^1\) is a symplectic manifold called the \textit{symplectic reduction} of \(M\) at 0. If the action is almost free, then the orbit space is an orbifold and one can still consider reduction. For actions of general compact Lie groups, one can show that, similar to the circle case, the level set is coisotropic and the symplectic complement of its tangent space is spanned by the fundamental vector fields of the action.
3.2 The Duistermaat-Heckman Theorem

Let $(M, \omega)$ be a symplectic manifold with a semifree Hamiltonian $S^1$-action. Suppose $0$ is a regular value of the moment map $\mu$. Since the critical values are isolated, we can find an $\epsilon > 0$ such that there are no critical values in the interval $(-\epsilon, \epsilon)$. The Duistermaat-Heckman theorem [DH82] relates the cohomology class of the reduced symplectic form on $M_0 = \mu^{-1}(0)/S^1$ with the class of the reduced form on $M_\delta = \mu^{-1}(\delta)/S^1$ for some $0 < |\delta| < |\epsilon|$. Specifically,

$$[\omega_\delta] = [\omega_0] + \delta \cdot u$$

where $u$ is the Chern class of the $S^1$-bundle $\mu^{-1}(0) \to M_0$.

One approach to the proof utilizes the equivariant coisotropic embedding theorem to get an equivariant symplectomorphism between $\mu^{-1}(-\epsilon, \epsilon)$ and $\mu^{-1}(0) \times (-\epsilon, \epsilon)$ with symplectic form

$$\sigma = \pi^*\omega_0 - d(t\alpha)$$

where $\pi : \mu^{-1}(0) \to M_0$ is the canonical projection, $\alpha$ is a connection form on $\mu^{-1}(0)$ and $t$ is a coordinate on $(-\epsilon, \epsilon)$. See Cannas da Silva [Ca01] for details. The coisotropic embedding theorem is originally due to Gotay [Got82] and can be thought of as a symplectic result since it relies on the Moser trick. There is another proof given by Audin [Au04] which we will outline here that uses only equivariant cohomology.

The map $j : \mu^{-1}(\epsilon) \to M$ is $S^1$-equivariant so induces a map

$$j \times_{S^1} 1 : \mu^{-1}(\epsilon) \times_{S^1} ES^1 \to M \times_{S^1} ES^1$$

where $u$ is the Chern class of the $S^1$-bundle $\mu^{-1}(0) \to M_0$. 

$$[\omega_\delta] = [\omega_0] + \delta \cdot u$$

where $u$ is the Chern class of the $S^1$-bundle $\mu^{-1}(0) \to M_0$. 

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$$\sigma = \pi^*\omega_0 - d(t\alpha)$$

where $\pi : \mu^{-1}(0) \to M_0$ is the canonical projection, $\alpha$ is a connection form on $\mu^{-1}(0)$ and $t$ is a coordinate on $(-\epsilon, \epsilon)$. See Cannas da Silva [Ca01] for details. The coisotropic embedding theorem is originally due to Gotay [Got82] and can be thought of as a symplectic result since it relies on the Moser trick. There is another proof given by Audin [Au04] which we will outline here that uses only equivariant cohomology.

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where $u$ is the Chern class of the $S^1$-bundle $\mu^{-1}(0) \to M_0$.
and a map of Weil models

\[ j_h^* : \Omega^*(M) \to \Omega^*(\mu^{-1}(\epsilon)). \]

Since the action is Hamiltonian, there is a closed, equivariant extension of \( \omega \) in \( \Omega^2(M) \) given by \( \tilde{\omega} = \omega \otimes 1 + D(\mu \otimes \theta) \) (2.3.3) where \( \theta \) is a connection 1-form for \( ES^1 \to BS^1 \) and \( \mu \) is the moment map. Since \( j_h^*(d\mu \otimes \theta) = 0 \),

\[ j_h^*(\tilde{\omega}) = \omega \otimes 1 + \mu \otimes D\theta, \]

so under the isomorphism \( \Omega^*_h(\mu^{-1}(\epsilon)) \simeq \Omega^*(M_{\text{red}}) \), we get

\[ \omega \otimes 1 + \mu \otimes D\theta \mapsto \omega_{\text{red}} + \epsilon \cdot \xi \]

where \( \xi \) is the Euler class of the \( S^1 \)-bundle \( \mu^{-1}(\epsilon) \to M_{\text{red}} \).

### 3.3 Symplectic Cuts and Symplectic Sums

The operation of symplectic cutting was introduced by Lerman [Le95]. Let \((M, \omega)\) be a semifree Hamiltonian \( S^1 \)-manifold with moment map \( \mu \). Suppose that 0 is a regular value of \( \mu \), so the level set \( \mu^{-1}(0) \) is an \( S^1 \)-bundle over \( M_0 \), the reduction at 0. The manifolds \( M_{\mu \leq 0} = \mu^{-1}(-\infty, 0] \) and \( M_{\mu \geq 0} = \mu^{-1}[0, \infty) \) have boundary \( \mu^{-1}(0) \). Taking the quotient of \( M_{\mu \leq 0} \), respectively \( M_{\mu \geq 0} \), by the \( S^1 \)-action on the boundary yields a closed manifold \( \tilde{M}_{\mu \leq 0} \), respectively \( \tilde{M}_{\mu \geq 0} \). In fact, they are semifree Hamiltonian \( S^1 \)-manifolds with maximum, respectively minimum, fixed point component \( M_0 \). Lerman’s construction defines the symplectic form on each half.
The inverse of symplectic cutting is known as *symplectic sum* and is due to Gompf [Gom95] and McCarthy-Wolfson [MW94]. Let \((M_1, \omega_1)\) and \((M_2, \omega_2)\) be two symplectic manifolds with codimension 2 symplectic submanifolds \(N_1 \approx N_2\). Suppose furthermore that the Euler classes of the normal bundles \(\nu N_1\) and \(\nu N_2\) satisfy \(e(\nu N_2) = -e(\nu N_1)\). Then we can glue \(M_1\) to \(M_2\) along the complements of the zero sections in \(\nu N_1\) and \(\nu N_2\) in such a way that when restricted to fibres is a symplectic inversion of the 2-dimensional annulus. This construction will not work in higher codimensions since there is no symplectic inversion of a \(2k\)-dimensional annulus for \(k > 1\), since such a map would yield a symplectic form on \(S^{2k}\).

### 3.4 The McDuff Construction

Let \((M, \omega)\) be a semifree non-Hamiltonian symplectic \(S^1\)-manifold and let \(X\) denote the fundamental vector field of the action. \(H^1(M; \mathbb{Z}) \simeq [M, S^1]\), so if \(i_X \omega\) is integral, then there is a smooth map \(\mu : M \to S^1\) with \(\mu^*(d\theta) = i_X \omega\) where \([d\theta]\) generates \(H^1(S^1; \mathbb{Z})\). Since a semifree action is symplectic iff \(L_X \omega = 0\), we can assume that \(\omega\) is integral and thus, that \(i_X \omega\) is integral. \(\mu\) is called an \(*S^1*-valued moment map* and was introduced by McDuff [Mc88].

**Remark 3.4.1.** A version of symplectic cutting can be accomplished on a manifold with a non-Hamiltonian action and \(S^1\)-valued moment map which will result in a Hamiltonian action. The action is lifted to a covering and ordinary symplectic cutting is done at two regular values covering the same value in \(S^1\). Details on lifting the action can be found in Ortega-Ratiu [OR05].

**Definition 3.4.2.** For a Hamiltonian \(S^1\)-manifold, call the fixed point components not corresponding to the extrema of the moment map *internal*. 

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Suppose $(M^{2n}, \omega)$ is a semifree Hamiltonian $S^1$-manifold with internal fixed point components $Z$ of codimension 4 and hence index 2. By McDuff [Mc88], the symplectic reductions at any regular value of the moment map $\mu$ are all diffeomorphic. Let $\tilde{M}$ denote their common diffeotype. Suppose $c$ is a critical value of $\mu$ corresponding to a fixed point component $Z$. There is $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon)$ contains no other critical values. Let $0 < \delta < \epsilon$. Also by McDuff [Mc88],

$$u_{c+\delta} - u_{c-\delta} = d_Z$$

(3.4.1)

where $u_{c+\delta}$, respectively $u_{c-\delta}$, is the first Chern class of the $S^1$-bundle $\mu^{-1}(c+\delta) \to \tilde{M}$, respectively $\mu^{-1}(c - \delta) \to \tilde{M}$, and $d_Z$ is the Poincaré dual of $Z$ considered as a submanifold of $\tilde{M}$. Note that this is a purely cohomological condition. If $M$ results from appropriately cutting the universal cover of some non-Hamiltonian action with codimension 4 fixed point set $F$, then we must have

$$\sum_{Z \subseteq F} d_Z = 0$$

(3.4.2)

as $Z$ runs over the components of $F$.

**Remark 3.4.3.** Let $H^2_+(\tilde{M})$ be the positive cone in $H^2(\tilde{M})$, namely, the subspace on which $\int_{\tilde{M}} \omega_{\text{red}}^{n-1} > 0$. By nondegeneracy, the variation of the reduced symplectic form must take place entirely in $H^2_+(\tilde{M})$. This fact, coupled with the Duistermaat- Heckman theorem, implies that any semifree non-Hamiltonian symplectic $S^1$-action with codimension 4 fixed point set results in a closed, piecewise linear curve in $H^2_+(\tilde{M})$.

**Remark 3.4.4.** By McDuff [Mc88], given a fixed point component $Z$ corresponding to a critical value $c$ of $\mu$, there is an $\epsilon > 0$ such that $Z$ must be a symplectic submanifold of $\tilde{M}$ for all reduced symplectic forms $\omega_{c+\delta}$ for $|\delta| < \epsilon$. The condition that a fixed
point component is symplectic implies

\[ \int_Z \omega_{\text{red}}^{n-2} = \int_{M_{\text{red}}} dZ \cdot \omega_{\text{red}}^{n-2} > 0 \]

by Poincaré duality. Since

\[ \int_{M_{\text{red}}} \left( \sum_{Z \subseteq F} dZ \right) \cdot \omega_{\text{red}}^{n-2} = 0, \]

no fixed point component \( Z \) will be symplectic for all reduced forms \( \omega_{\text{red}} \). In fact, given any \( \xi \in H^2(M_{\text{red}}) \),

\[ \int_{M_{\text{red}}} \left( \sum_{Z \subseteq F} dZ \right) \cdot \xi = \sum_{Z \subseteq F} \int_Z \xi |_{Z} = 0. \]

**Example 3.4.5** (McDuff [Mc88]). Let \( B = T^4 \) with coordinates \((x^1, x^2, x^3, x^4)\), \( L_{ij} = T^2 \subset B \) with coordinates \((x^i, x^j)\) and \( \sigma_{ij} = dx^i dx^j \). \( B \) is the diffeotype of \( M_{\text{red}} \) and the \( L_{ij} \) are the internal fixed point components. Let \( c_s \) be the Chern class of the \( S^1 \)-bundle \( \mu^{-1}(s) \to B \) for a regular value \( s \), \( Z \) be a fixed point component, and \( dZ \) be the Poincaré dual of \( Z \) in \( B \). The original data for the McDuff construction is a family of symplectic forms \( \tau_s \) on \( B \).

\[ \begin{align*}
& s \in [0,1) \quad \tau_s = K\sigma_{12} + K\sigma_{34} + 2\sigma_{13} + 2\sigma_{42} \quad c_s = 0 \\
& s = 1 \quad Z = L_{13} \quad dZ = [\sigma_{42}] \\
& s \in (1,2) \quad \tau_s = K\sigma_{12} + K\sigma_{34} + 2\sigma_{13} + (3-s)\sigma_{42} \quad c_s = -[\sigma_{42}] \\
& s = 2 \quad Z = L_{42} \quad dZ = [\sigma_{13}] \\
& s \in (2,5) \quad \tau_s = K\sigma_{12} + K\sigma_{34} + (4-s)\sigma_{13} + (3-s)\sigma_{42} \quad c_s = -[\sigma_{13} + \sigma_{42}] \\
& s = 5 \quad Z = L_{31} \quad dZ = -[\sigma_{42}] \\
& s \in (5,6) \quad \tau_s = K\sigma_{12} + K\sigma_{34} + (4-s)\sigma_{13} - 2\sigma_{42} \quad c_s = -[\sigma_{13}] \\
& s = 6 \quad Z = L_{24} \quad dZ = -[\sigma_{13}] \\
& s \in (6,7] \quad \tau_s = K\sigma_{12} + K\sigma_{34} - 2\sigma_{13} - 2\sigma_{42} \quad c_s = 0
\]
We can double the McDuff data to obtain a closed, piecewise linear curve in the space of closed 2-forms on $B$.

\[ s \in [7, 8) \quad \tau_s = K\sigma_{12} + K\sigma_{34} - 2\sigma_{13} - 2\sigma_{42} \quad c_s = 0 \]
\[ s = 8 \quad Z = L_{31} \quad dZ = -[\sigma_{42}] \]
\[ s \in (8, 9) \quad \tau_s = K\sigma_{12} + K\sigma_{34} - 2\sigma_{13} - (10 - s)\sigma_{42} \quad c_s = [\sigma_{42}] \]
\[ s = 9 \quad Z = L_{24} \quad dZ = -[\sigma_{13}] \]
\[ s \in (9, 12) \quad \tau_s = K\sigma_{12} + K\sigma_{34} - (11 - s)\sigma_{13} - (10 - s)\sigma_{42} \quad c_s = [\sigma_{13} + \sigma_{42}] \]
\[ s = 12 \quad Z = L_{13} \quad dZ = [\sigma_{42}] \]
\[ s \in (12, 13) \quad \tau_s = K\sigma_{12} + K\sigma_{34} - (11 - s)\sigma_{13} + 2\sigma_{42} \quad c_s = [\sigma_{13}] \]
\[ s = 13 \quad Z = L_{42} \quad dZ = [\sigma_{13}] \]
\[ s \in (13, 14] \quad \tau_s = K\sigma_{12} + K\sigma_{34} + 2\sigma_{13} + 2\sigma_{42} \quad c_s = 0 \]

It remains to find a compatible semifree Hamiltonian $S^1$-manifold $(M^6, \omega)$ with moment map $\mu : M \to [0, 14]$. That this can be done for codimension 4 fixed point components is due to Guillemin-Sternberg [GS89] and is detailed in Gonzalez [Gon05]. A semifree non-Hamiltonian symplectic $S^1$-manifold results from identifying the boundary components $\mu^{-1}(0)$ and $\mu^{-1}(14)$.

**Remark 3.4.6.** The McDuff data can be thought of as the cohomological part of the construction. The symplectic part, that is to say, the part which utilizes some variant of the Moser trick, lies in the actual construction of the semifree Hamiltonian $S^1$-manifold $(M^6, \omega)$. In fact, over an interval $I$ of regular values of the moment map $\mu$, McDuff [Mc88] shows that $\omega$ is determined, up to $S^1$-equivariant symplectomorphism which leaves the level sets of $\mu$ invariant, by a family of reduced forms $\tau_s$ where $s$ varies over $I$, and the proof of this uses the Moser trick.
3.5 A Result on Almost Complex $S^1$-Actions

We now state and prove our main technical result which gives a cohomological con-
straint on McDuff-type constructions in higher dimensions. Given a semifree $S^1$-
action on an almost complex manifold $M$ which preserves the almost complex struc-
ture and has fixed point set $F$, recall that there is an equivariant splitting of the
normal bundle $\nu = \nu^+ \oplus \nu^-$ of $F$ into $(\pm 1)$-eigenbundles of the $S^1$-representation.

**Theorem 3.5.1.** Let $M^{2n}$ be a closed almost complex manifold with a semifree $S^1$-
action preserving the almost complex structure. Suppose that the fixed point compo-
nents $F_i^{2k}$ are of codimension 4 and index 2. Let $c_i^\pm$ be the first ordinary Chern class
of the $(\pm 1)$-eigenbundle $\nu_i^\pm$ of the normal bundle $\nu_i$ of the fixed point component $F_i$,
$c_j(F_i)$ be the $j^{th}$ ordinary Chern class of $F_i$ for $0 \leq j \leq k = n - 2$, and $[F_i]$ be the
fundamental class of $F_i$. Then we have

\[
\sum_i \left( c_k(F_i) + c_{k-1}(F_i)(c_i^+ + c_i^-) \right) [F_i] = 0, \tag{3.5.1}
\]

\[
\sum_i \left( c_{k-1}(F_i)(c_i^+ - c_i^-) + c_{k-2}(F_i)((c_i^+)^2 - (c_i^-)^2) \right) [F_i] = 0, \tag{3.5.2}
\]

and

\[
\sum_i \left( c_{k-2}(F_i)((c_i^+)^2 - c_i^+ c_i^- + (c_i^-)^2) + c_{k-3}(F_i)((c_i^+)^3 + (c_i^-)^3) \right) [F_i] = 0. \tag{3.5.3}
\]
Proof. Let \( c(F_i) \) be the ordinary total Chern class of \( F_i \) and \( c(\nu_i) \) be the equivariant total Chern class of \( \nu_i \). Since \( F_i \) is of codimension 4 and index 2, for \( \eta = c(M_{S^1})/c(BS^1) \) (2.6.4) we have

\[
\eta|_{F_i} = c(F_i) \cdot c(\nu_i) \quad = c(F_i)(1 + t + c_i^+)(1 - t + c_i^-) \quad = c(F_i)(1 + (c_i^+ + c_i^-) + e(\nu_i))
\]

where

\[
e(\nu_i) = (t + c_i^+)(-t + c_i^-) = -t^2 + (c_i^- - c_i^+)t + c_i^+ c_i^-.
\]

Localization (2.4.1) gives

\[
\chi(M) = \int_M c(M) = \int_M \eta|_M = \sum_i \int_{F_i} \frac{\eta|_{F_i}}{e(\nu_i)} = \sum_i \chi(F_i) + \sum_i \int_{F_i} \frac{c(F_i)(1 + (c_i^+ + c_i^-))}{e(\nu_i)}.
\]

Furthermore,

\[
\frac{1}{e(\nu_i)} = -\frac{1}{t^2} \left( 1 + \left( \frac{c_i^- - c_i^+}{t} + \frac{c_i^+ c_i^-}{t^2} \right) + \cdots + \left( \frac{c_i^- - c_i^+}{t} + \frac{c_i^+ c_i^-}{t^2} \right)^k \right)
\]

and

\[
\chi(M) = \sum_i \chi(F_i),
\]

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so we get

\[
\sum_i \int_{F_i} c(F_i)(1 + (c_i^+ + c_i^-)) \left( 1 + \left( \frac{c_i^- - c_i^+}{t} + \frac{c_i^+ c_i^-}{t^2} \right) + \cdots + \left( \frac{c_i^- - c_i^+}{t} + \frac{c_i^+ c_i^-}{t^2} \right)^k \right)
= 0 \in \mathbb{Q}(t).
\]

The equations (3.5.1), (3.5.2) and (3.5.3) follow from vanishing of the constant part, the coefficient of \( \frac{1}{t} \) and the coefficient of \( \frac{1}{t^2} \), respectively.

**Remark 3.5.2.** We only have to consider the coefficients of \( \frac{1}{t^l} \) for \( l \leq k \) since higher powers of \( \frac{1}{t} \) have coefficients of degree greater than \( 2k \). In general, we obtain the relation

\[
\sum_i \left( c_{k-i}(F_i) \left( \frac{(c_i^+)^{l+1} - (-c_i^-)^{l+1}}{c_i^+ + c_i^-} \right) + c_{k-i-1}(F_i)((c_i^+)^{l+1} - (-c_i^-)^{l+1}) \right) [F_i] = 0.
\]

### 3.6 \( S^1 \)-Actions on 6-Manifolds with Fixed Surfaces

Let \( (M^6, \omega) \) be a closed semifree non-Hamiltonian symplectic \( S^1 \)-manifold with fixed point components \( F_i \) of dimension 2. We can use symplectic cutting on the universal cover \( (\tilde{M}^6, \tilde{\omega}) \), and by appropriate scaling of \( \tilde{\omega} \), we can make the lifted form integral and thus make all the critical values of the moment map on the cut manifold integral. We will, by abuse of notation, denote the closed semifree Hamiltonian \( S^1 \)-manifold thus constructed as \( (M, \omega) \), the corresponding moment map as \( \mu \) and the internal fixed point components as \( F_i \). The fixed point components \( F_{\min} \) and \( F_{\max} \) corresponding to the extreme values of \( \mu \) are diffeomorphic to the symplectic reduction \( M_{\text{red}} \) at any regular value of \( \mu \) since the internal components are of codimension 4 and hence of index 2. See McDuff [Mc88]. Furthermore, the Chern class \( c_{\min} \) of the \( S^1 \)-bundle \( \mu^{-1}(\min + \epsilon) \to M_{\text{red}} \) must be the same as the Chern class \( c_{\max} \) of the \( S^1 \)-bundle
\( \mu^{-1}(max - \epsilon) \rightarrow M_{\text{red}} \). The equivariant Euler classes of the normal bundles \( \nu_{\text{min}} \) of \( F_{\text{min}} \) and \( \nu_{\text{max}} \) of \( F_{\text{max}} \) are

\[
e(\nu_{\text{min}}) = t + c, \\
e(\nu_{\text{max}}) = -e(\nu_{\text{min}})
\]

where \( t \) is a generator of \( H^*(BS^1) \) and \( c = c_{\text{min}} = c_{\text{max}} \). Denote the fundamental class of \( F_i \) by \([F_i]\) and let

\[c_i^\pm = c_1(\nu_i^\pm)[F_i]\]

be the first Chern numbers of the corresponding line bundles. The equivariant Euler class of \( \nu_i \) is

\[
e(\nu_i) = (t + c_i^+ u_i)(-t + c_i^- u_i) \\
= -t^2 + (c_i^- - c_i^+) tu_i
\]

where \( u_i \in H^2(F_i) \) satisfies \( u_i[F_i] = 1 \). The total Chern class of \( F_i \) is

\[c(F_i) = 1 + \chi(F_i) u_i.\]

For \( \eta = c(M_{S^1})/c(BS^1) \) (2.6.4) we have

\[
\eta|_{F_{\text{min}}} = c(F_{\text{min}})(1 + e(\nu_{\text{min}})), \\
\eta|_{F_{\text{max}}} = c(F_{\text{max}})(1 + e(\nu_{\text{max}})), \\
\eta|_{F_i} = c(F_i)(1 + (c_i^+ + c_i^-) u_i + e(\nu_i)),
\]
\[ \chi(M) = \int_M c(M) \]
\[ = \int_M \frac{\eta|_{F_{\min}}}{e(\nu_{\min})} + \int_{F_{\max}} \frac{\eta|_{F_{\max}}}{e(\nu_{\max})} + \sum_i \int_{F_i} \frac{\eta|_{F_i}}{e(\nu_i)} \]
\[ = \int_{F_{\min}} \frac{c(F_{\min})(1 + e(\nu_{\min}))}{e(\nu_{\min})} + \int_{F_{\max}} \frac{c(F_{\max})(1 + e(\nu_{\max}))}{e(\nu_{\max})} + \sum_i \int_{F_i} \frac{\eta|_{F_i}}{e(\nu_i)} \]
\[ = \chi(F_{\min}) + \chi(F_{\max}) + \sum_i \chi(F_i) + \sum_i \int_{F_i} \frac{c(F_i)(1 + (c_i^+ + c_i^-)u_i)}{e(\nu_i)} \]

and so
\[ \sum_i \int_{F_i} \frac{c(F_i)(1 + (c_i^+ + c_i^-)u_i)}{e(\nu_i)} = 0 \]

since
\[ \chi(M) = \chi(F_{\min}) + \chi(F_{\max}) + \sum_i \chi(F_i). \]

We have basically duplicated the method of proof of Theorem 3.5.1 in the case of a 6-manifold and thus have analogues of (3.5.1) and (3.5.2).

**Remark 3.6.1.** The preceding calculations are not identical to those of the proof of Theorem 3.5.1 since \( F_{\min} \) and \( F_{\max} \) are of codimension 2 in \( M \). We have used the non-Hamiltonian version of symplectic cutting which is briefly described in Remark 3.4.1, however, localization at \( \eta \) yields the same result for the uncut non-Hamiltonian manifold since
\[ \chi(M_{\text{cut}}) = \chi(M_{\text{uncut}}) + \chi(F_{\min}) + \chi(F_{\max}). \]

**Proposition 3.6.2.** Let \( M^6 \) be a closed semifree non-Hamiltonian symplectic \( S^1 \)-manifold with fixed point set \( F \) consisting of surfaces \( F_i \). Let \( c_i^\pm \) be the first Chern
numbers of the \((\pm1)\)-eigenbundles of the \(S^1\)-representation on the normal bundle \(\nu_i\) of \(F_i\). Then

\[
\sum_i \left( c_i^+ + c_i^- + \chi(F_i) \right) = 0 \tag{3.6.1}
\]

and

\[
\sum_i (c_i^+ - c_i^-) = 0. \tag{3.6.2}
\]

We can obtain a sharper result than (3.6.1) when \(c_1(M_{\text{red}}) = 0\). Since \(M_{\text{red}}\) is 4-dimensional, another result of McDuff [Mc88] says that the normal bundle of \(F_i\) in \(M_{\text{red}}\) is

\[\nu_{i}^{\text{red}} = \nu_i^+ \otimes \nu_i^-\]

The Poincaré dual \(d_i\) of \(F_i\) in \(M_{\text{red}}\) is the Thom class of the normal bundle \(\nu_i^{\text{red}}\) and thus satisfies

\[
d_i|_{F_i} = e(\nu_i^{\text{red}})
\]

\[
= c_1(\nu_i^+ \otimes \nu_i^-) \tag{3.6.3}
\]

\[
= (c_i^+ + c_i^-) u_i.
\]

See Bott-Tu [BT82] for details on the Poincaré dual of a closed submanifold. An embedded \(j\)-holomorphic curve \(C\) in an almost complex 4-manifold \(X\) must satisfy the adjunction formula

\[
C \cdot C - \langle c_1(X), C \rangle + 2 = 2g \tag{3.6.4}
\]

where \(C \cdot C\) is the self-intersection of \(C\), \(\langle \cdot , \cdot \rangle\) is the pairing between cohomology and homology, and \(g\) is the genus of \(C\). See McDuff-Salamon [MS04]. The formula is
established by observing that

\[ C \cdot C = \int_X d_C \cdot d_C = \int_C e(\nu_C) \]  

and

\[ \langle c_1(X), C \rangle = \int_X c_1(X) \cdot d_C = \int_C e(C) + e(\nu_C) \]  

where \( d_C \) is the Poincaré dual of \( C \) in \( X \) and \( \nu_C \) is the normal bundle of \( C \) in \( X \).

**Theorem 3.6.3.** Let \( M^6 \) be a closed semifree non-Hamiltonian symplectic \( S^1 \)-manifold with fixed point set \( F \) consisting of surfaces \( F_i \). Let \( c_i^\pm \) be the first Chern numbers of the \((\pm 1)\)-eigenbundles of the \( S^1 \)-representation on the normal bundle \( \nu_i \) of \( F_i \). If \( c_1(M_{\text{red}}) = 0 \), then for each fixed point component \( F_i \), we have

\[ c_i^+ + c_i^- + \chi(F_i) = 0. \]  

**Proof.** Each \( F_i \) is a symplectic submanifold, and hence \( j \)-holomorphic curve, of \( M_{\text{red}} \) for an appropriate reduced form \( \omega_{\text{red}} \). See Remark 3.4.4. By (3.6.3) and (3.6.5),

\[ F_i \cdot F_i = c_i^+ + c_i^- . \]

Since \( \chi(F_i) = 2 - 2g \) where \( g \) is the genus of \( F_i \), the result follows from the adjunction formula (3.6.4).

**Remark 3.6.4.** \( \sum_i d_i = 0 \) by (3.4.2), so (3.6.6) gives another proof of (3.6.1).
We obtain additional results concerning the self-intersections of the fixed surfaces $F_i$ thought of as submanifolds of a fixed copy of $M_{\text{red}}$. Recall that the $F_i$ cannot all be symplectic by Remark 3.4.4.

**Proposition 3.6.5.** Let $M^6$ be a closed semifree non-Hamiltonian symplectic $S^1$-manifold with fixed surfaces $F_i$. Then

$$\sum_i \chi(F_i) = 2 \sum_{i < j} (F_i \cdot F_j)$$

when the $F_i$ are thought of as submanifolds of $M_{\text{red}}$. In particular, when $F_i \approx S^2$,

$$\sum_{i < j} (F_i \cdot F_j) = \# \text{ of fixed 2-spheres}. \quad (3.6.9)$$

**Proof.** $\sum_i d_i = 0$ by (3.4.2), so by (3.6.3) and (3.6.5), we have

$$\int_{M_{\text{red}}} \left( \sum_i d_i \right)^2 = \sum_i (F_i \cdot F_i) + 2 \sum_{i < j} (F_i \cdot F_j)$$

$$= \sum_i \left( c^+_i + c^-_i \right) + 2 \sum_{i < j} (F_i \cdot F_j)$$

$$= 0.$$

The result follows from (3.6.1). \qed

**Remark 3.6.6.** The proofs of the results of this section are similar in spirit to those of Li [Li03b].

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3.7 \(S^1\)-Actions with Isolated Fixed Points

The following Tolman-Weitsman result [TW00] on semifree \(S^1\)-actions on \(M^{2n}\) with isolated fixed points is recovered readily by localizing at \(c(M_{S^1})/c(BS^1)\).

**Theorem 3.7.1** (Tolman-Weitsman). Let \(M^{2n}\) be a connected semifree symplectic \(S^1\)-manifold. Suppose the action has isolated fixed points. Then the action is Hamiltonian and the number \(N_k\) of fixed points of index \(2k\) satisfies \(N_k = \binom{n}{k}\), whence \(\chi(M) = 2^n\).

**Proof.** Let \(\eta = c(M_{S^1})/c(BS^1)\) (2.6.4). Then

\[
\eta|_p = c(p) \cdot c(\nu_p) = (1 + t)^{n-k}(1 - t)^k
\]

and

\[
e(\nu_p) = c_n(\nu_p) = (-1)^k t^n
\]

where \(t\) is a degree 2 generator of \(H^*(BS^1)\). Localization (2.4.1) gives

\[
\chi(M) = \int_M c(M)
= \int_M \eta|_M
= \sum_{p\in F} \frac{\eta|_p}{e(\nu_p)}
= \sum_{p\in F} \frac{(1 + t)^{n-k}(1 - t)^k}{(-1)^k t^n}
= \frac{1}{t^n} \left( \sum_{k=0}^{n} (-1)^k N_k (1 + t)^{n-k}(1 - t)^k \right)
\]
Evaluation at $t = 1$ yields

\[ 2^n N_0 = \chi(M) = \# \text{ of fixed points}. \]

Thus $N_0 \neq 0$, so the moment map has an absolute minimum and therefore the action is Hamiltonian. Furthermore, since the fixed point component of the minimum is connected, $N_0 = 1$, so

\[ \chi(M) = 2^n. \]

Applying $\frac{d}{dt}\bigg|_{t=1}$ to the equation

\[ \chi(M)t^n = \sum_{k=0}^{n} (-1)^k N_k (1 + t)^{n-k}(1 - t)^k \]

gives

\[ n\chi(M) = 2^{n-1}N_1 + n2^{n-1}N_0 \]

from which $N_1 = n$ follows. Continuing in this fashion, we get

\[ N_k = \binom{n}{k}. \]
Chapter 4

Cohomological Conditions

4.1 A Todd Genus Condition

A remarkable result of Feldman [Fe01] relates the Todd genus $Td(M)$ of a symplectic manifold $M$ to Hamiltonian $S^1$-actions on $M$. Assume throughout that $M$ is closed and connected.

**Theorem 4.1.1** (Feldman). Suppose a symplectic $S^1$-manifold $M$ has isolated fixed points. Then $Td(M) = 1$ if the action is Hamiltonian and $Td(M) = 0$ if the action is non-Hamiltonian. Any action on a symplectic manifold with $Td(M) > 0$ is Hamiltonian.

Feldman’s result follows from the formula

$$
\chi_y(M) = \sum_s (-y)^{d_s} \chi_y(M_s)
$$

where $M_s$ is a component of the fixed point set and $d_s$ is the number of negative weights of the $S^1$-representation on the normal bundle of $M_s$, namely, half the index. $\chi_y(M)$ is the Hirzebruch $\chi_y$-genus of $M$ and $Td(M) = \chi_y(M) |_{y=0}$. 

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We observe that for a Hamiltonian action $Td(M) = Td(M_{\text{min}})$ where $M_{\text{min}}$ is the fixed point component corresponding the minimum of the moment map. Reversing the $S^1$-action interchanges $M_{\text{min}}$ and $M_{\text{max}}$ so

$$Td(M) = Td(M_{\text{min}}) = Td(M_{\text{max}}).$$

(4.1.1)

Lemma 4.1.2. Let $M$ be a semifree Hamiltonian $S^1$-manifold. Then

$$Td(M) = Td(M_{\text{min}}) = Td(M_{\text{max}}) = Td(M_{\text{red}}).$$

Proof. Apply symplectic cutting to any regular value of the moment map and use (4.1.1).

This is similar to a result of Li [Li03a] which states that for a semifree Hamiltonian $S^1$-manifold,

$$\pi_1(M) = \pi_1(M_{\text{min}}) = \pi_1(M_{\text{max}}) = \pi_1(M_{\text{red}}).$$

Theorem 4.1.3. Let $M$ be a symplectic manifold, $N$ be a closed symplectic submanifold, and $\tilde{M}$ be the blow up of $M$ along $N$. Then

$$Td(\tilde{M}) = Td(M).$$

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Proof. By Guillemin-Sternberg [GS89], we can consider $M$ as the reduction $X_{-\varepsilon}$ at $-\epsilon < 0$ of a semifree Hamiltonian $S^1$-manifold $X$ where 0 is a critical value of the moment map with corresponding fixed point component $N$ of (co)index 2. The reduction $X_\epsilon$ at $\epsilon > 0$ is the blow up $\tilde{M}$ of $M$ along $N$. The result follows from Proposition 4.1.1.

**Proposition 4.1.4.** Let $M$ be a symplectic $S^1$-manifold with $c_1(M) = 0$. Then

$$Td(M) = 0.$$  

**Proof.** Recall that a manifold $M$ has a spin$^c$-structure if the second Stiefel-Whitney class $w_2(M)$ is the mod 2 reduction of an integral class $c \in H^2(M)$. Every symplectic manifold $M$ has a spin$^c$-structure. For any spin$^c$-manifold $M$,

$$td(M) = \hat{a}(M)e^{c_1(M)/2}$$

where $td(M)$ is the Todd class of $M$ and $\hat{a}(M)$ is the Dirac $\hat{A}$-class of $M$. If $c_1(M) = 0$, then $M$ is a spin $S^1$-manifold, so

$$Td(M) = \hat{A}(M) = 0$$

by a result of Atiyah-Hirzebruch [AH70].

**Definition 4.1.5.** A Calabi-Yau manifold is a Kähler manifold with $c_1 = 0$.

**Corollary 4.1.6.** If $M$ is a Calabi-Yau manifold equipped with a Hamiltonian $S^1$-action, then $Td(M_{\text{min}}) = Td(M_{\text{max}}) = 0$. In particular, $M_{\text{min}}$ and $M_{\text{max}}$ cannot be points.
4.2 \( \omega|_{\pi_2(M)} = 0 \) and Related Conditions

Let \((M, \omega)\) be a symplectic manifold. Let \(\xi\) be the fundamental vector field of a symplectic \(S^1\)-action on \(M\). Then \(L_\xi\omega = 0\), so by the Cartan homotopy formula (1.3.1), \(i_\xi\omega\) is closed. The action is Hamiltonian iff \(i_\xi\omega\) is exact. Cohomologically, \(M\) is a Hamiltonian \(S^1\)-manifold iff \(\omega\) has a closed equivariant extension in \(H^*_S(M)\) iff \(d_2\omega = 0\) in the Serre spectral sequence of the Borel construction \(M_{S^1}\). It is known that the Chern classes of \(M\) survive to \(E_\infty\) in the Serre spectral sequence of \(M_{S^1}\), so an action on a monotone symplectic manifold, \(k\omega = c_1\), must be Hamiltonian. Note that the existence of fixed points is not needed.

Hamiltonian \(S^1\)-actions must have fixed points since the image of the moment map will be a closed interval whose endpoints will be critical values. In fact, the moment map is a Morse-Bott function so the Serre spectral sequence of \(M_{S^1}\) collapses at \(E_2\), namely, \(d_2(\omega) = 0\) implies \(E_2 = E_\infty\).

**Proposition 4.2.1** (Ono [On92]). Let \((M, \omega)\) be a semifree non-Hamiltonian symplectic \(S^1\)-manifold. If \(\omega|_{\pi_2(M)} = 0\), then the fixed point set \(F = \emptyset\).

**Proof.** (Allday [Al06]) Suppose \(F \neq \emptyset\). Then there is a nontrivial element \(\alpha \in \pi_1(M - F)\) which is not in the image of \(\pi_1(i_s)\) where \(i_s : \mu^{-1}(s) \to M - F\) is the inclusion of the level set of any regular value \(s\). Joining the image of a representative of \(\alpha\) to a fixed point of the action by a path in \(M - F\) yields \(S^1 \vee [0, 1]\). Using the circle action, we obtain \(T^2\) with a 2-disk sewed onto a meridian. The resulting space is homotopic to \(S^2 \vee S^1\) and thus represents a nontrivial element \(\hat{\alpha} \in \pi_2(M)\) satisfying \(\omega(\hat{\alpha}) \neq 0\), a contradiction. \(\square\)

Consider the condition \((k\omega - c_1(M))|_{\pi_2(M)} = 0\). \(M\) is called spherically monotone. Let \(X = S^2 \times T^2\) and let \(S^1\) act on the second factor of \(T^2\) in the standard fashion.
By the long exact homotopy sequence of $S^2 \to X \to T^2$, $\pi_2(X) = \pi_2(S^2)$, so given an element $\alpha \in \pi_2(X)$, $\alpha = i_*(\hat{\alpha})$ where $\hat{\alpha} \in \pi_2(S^2)$ and $i : S^2 \to X$ is the inclusion into the first factor. Let $1 \in \pi_2(X)$ be $i_*(\text{id}_{S^2})$. Then $\alpha = n \cdot 1$ so

$$(k\omega - c_1(X))(\alpha) = kn\omega(1) - 2n = 0$$

for $k = \frac{2}{\omega(1)}$ which is independent of $n$. On the other hand, if $S^1$ acts on the first factor, then $X$ has fixed points, so spherical monotonicity says nothing about the existence of fixed points. However, Allday [Al06] has shown that if $M$ is spherically monotone and has a fixed point, then the action is Hamiltonian.

Remark 4.2.2. Another way of seeing that $X = S^2 \times T^2$ is spherically monotone is to note that $c_1(X) = c_1(S^2)$. 

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