CORE AND NO-TREAT EQUILIBRIUM IN TOURNAMENT GAMES WITH EXTERNALITIES

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Abstract. We consider a situation where coalitions are formed to divide a resource. As in real life, the value of a payoff to a given agent is allowed to depend on the payoff to other agents with whom he shares a common interest. There are various notions of equilibrium for this type of game, including the core and no-treat equilibrium. These stabilities may exist or not, depending on the power structure and the rule for allocating the resource. It is shown that under certain conditions, the no-treat equilibrium can exist even though the core is empty.

1. Introduction

Today, game theory is increasingly being used to model interactions in social science, political science, psychology, and especially economics. But it is actually a field of applied mathematics, one that attempts to mathematically capture behavior in strategic situations in which an individual's success in making choices depends on the choices of others.

The reasons I chose game theory for my master's project were that I got interested in when I studied Game Theory in an undergraduate political science class, and that it is closely related to our everyday experience.

Although Game Theory is used in a lot of disciplines, it requires many advanced mathematical techniques, such as analysis, linear algebra, abstract algebra, and so on. However, game theory is often applied in these disciplines without using those advanced techniques. So I would like to extend my mathematical skills to consider one of the models in economics which is coalition-formation.

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2. Definitions and concepts

We are going to analyze agents who are looking to maximize their share of a divisible resource by being a singleton or forming coalitions.

Let M be a divisible resource, say money. Let N be a set of agents $N = \{1, 2, ..., n\}$, and each agent has an additive preference on his share of money and other agents' shares. Each agent has a power described as $\pi_1, \pi_2, ..., \pi_n$ with $\pi_i \geq 0$ and $\sum_{i=1}^{n} \pi_i = 1$. Also, without loss of generality, we assume that $\sum_{i \in S} \pi_i \neq \sum_{i \in T} \pi_i$ for all $S \neq T$, so we do not have any ties.

A *coalition* is a group of agents which they might want to form with the other agent(s) to win the game. The *winning coalition* is the coalition $S \subseteq N$ with $\sum_{i \in S} \pi_i$ maximum.

A *partition* is a set of coalitions and/or singletons, and each agent belongs to one of the elements in the partition.

Let ζ be a function that specifies the allocations of the resource across the winning agents, That is, for any agent $i \in S \subseteq N$, $\zeta_i(S)$ is the allocation of the money to agent i with $\sum_{i=1}^{n} \zeta_i(S) = M$ when coalition S is winning. We assume that ζ is *cross-monotonic* on the size of the coalition, that is $\zeta_i(S) > \zeta_i(T)$ for $i \in S \subset T$.

Another concept which is very important for this project is *externality*, which is basically a situation in which each agent cares not only about himself, but also possibly cares about the other agents. Those relationships are represented by an $n \times n$ matrix for n agents, with entries M_{ij} representing the externality that agent j imposes on agent i. We assume that $M_{ii} = 1$ and $\sum_{i \neq j} |M_{ij}| < 1$ so that any agent's altruism does not exceed their own self-interest.

We are going to consider two rules for dividing money to agents in the winning coalition, which are equal sharing and proportional sharing. Let me introduce these two sharing rules. Let S be the winning coalition.

(1) Equal sharing is given by

$$\zeta_i(S) = \begin{cases} \dfrac{M}{|S|} & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

So under equal sharing, all agents in the winning coalitions share the same amount of the resource.

(2) Proportional sharing is given by

$$\zeta_i(S) = \begin{cases} \dfrac{\pi_i}{\sum_{j \in S} \pi_j} M & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

So under proportional sharing, each agent's share depends on his power and the total power in the winning coalition.

There are other sharing rules besides above two rules, but we only consider these two in this project.

Definition 1. For the *payoff function*, we define $U_i(x_1, x_2, ..., x_n) = \sum_j M_{ij} x_j$ where M_{ij} is the *externality* that agent j imposes on agent i.

Definition 2. For a partition Π, the *net utility* to agent i is $v_i(\Pi) = \sum_{j \in S^*} M_{ij} \zeta_j(S^*)$ where S^* is the coalition in Π with the largest power.

Definition 3. The *core* is the set of allocations that cannot be improved by a coalition of agents. That is, a partition Π is in the core if there is no subset $S \subseteq N$ such that $v_i(\Pi - S, S) > v_i(\Pi)$ for all $i \in S$, so that all agents in the winning coalition are better off by being in that coalition in Π. (Note that $\Pi - S = \Pi$ restricted $N - S$ for any S.)

Definition 4. Similarly to the core, we define an alternative notion, no-treat equilibrium, call it NTE, under which agents can react to a deviation in a way that harms the agents who originally deviated. So Π is NTE if whenever $S \subseteq N$ is such that $v_i(\Pi - S, S) > v_i(\Pi)$ for all $i \in S$, then there exists $T \subseteq N - S$ such that $v_i(\Pi - (S \cup T), T, S) > v_i(\Pi - S, S)$ for all $i \in T$ and $v_i(\Pi - (S \cup T), T, S) < v_i(\Pi)$ for some $i \in S$.
Note that NTE is a relaxation of the core, that is if a partition Π is in the core, then Π is in NTE. On the other hand, we will see below that there can be NTE partition where the core is empty.

Definition 5. The *minimally winning coalition* is a winning coalition $S^* \subseteq N$ satisfying $\sum_{i \in S^*} \pi_i > 1/2 > \sum_{i \in S^* - \{j\}} \pi_i$ for all $j \in S^*$.

Definition 6. The *minimally winning coalition of the minimal size* is a minimally winning coalition $S^* \subseteq N$ satisfying $\sum_{i \in S^*} \pi_i < \sum_{i \in S} \pi_i$ for all $S \subseteq N$ with S minimally winning. This is used under the equal sharing.

Definition 7. The *minimally winning coalition of the minimal weight* is minimally winning coalition $T^* \subseteq N$ such that $\sum_{i \in T^*} \pi_i < \sum_{i \in T} \pi_i$ for all $T \subseteq N$ minimally winning. This is used under the proportional sharing.

With these definitions, what we are going to consider is the following;

- Does there always exist a core-stable partition?
- Can we characterize the set of rules that have a core-stable partition?
- Does there always exist a NTE-stable partition?

- Can we characterize the set of rules that have a NTE-stable partition?

3. No externalities

Observe that, if there are no externalities, then as we increase the size of a coalition any winning agent is worse off (because his share decreases), and agents who are not winning would prefer to be winning (because their net utility is zero when losing). This will also be true in the case with externalities under weak conditions.

3.1. **Equally shared case.**

For equally shared case, we cannot guarantee that the core and NTE always exist. Also, we cannot guarantee that if the core is empty, then NTE does not exist. Here is an example where the core does not exist, but NTE exists.

Example 8. Consider the game with $N = \{1, 2, 3, 4, 5\}$ and $\pi = (.36, .34, .12, .10, .08)$. Then each of the following are partitions with minimally winning coalitions. Now, consider whether these minimally winning coalitions are in the core or NTE.

- $(\{1, 2\}, \{3, 4, 5\})$ is NTE. (Agent 1 does not want to deviate because then agent 2 can form a coalition with agent 3,4, and 5.) But this partition is not in the core because agent 1 can deviate.
- $(\{1, 3, 4\}, \{2, 5\})$ is not NTE. (Agent 1 is not better of in this coalition because forming a coalition with 2 increases his share.) Therefore, this partition is not in the core also.
- $(\{1, 4, 5\}, \{2, 3\})$ is not NTE. (Agent 1 is not better of in this coalition because forming a coalition with 2 increases his share.) There for this partition is not in the core also.
- $(\{2, 3, 4\}, \{1, 5\})$ is not NTE. (Agent 2 is not better of in this coalition because forming a coalition with 1 increases his share.) Therefore, this partition is not in the core also.
- $(\{2, 4, 5\}, \{1, 3\})$ is not NTE. (Agent 2 is not better of in this coalition because forming a coalition with 1 increases his share.) Therefore, this partition is not in the core also.
- $(\{1, 3, 5\}, \{2, 4\})$ is not NTE. (Agent 1 is not better of in this coalition because forming a coalition with 2 increases his share.) Therefore, this partition is not in the core also.
- $(\{2, 3, 5\}, \{1, 4\})$ is not NTE. (Agent 2 is not better of in this coalition because forming a coalition with 1 increases his share.) Therefore, this partition is not in the core also.

As we have seen above, this game has an NTE, which is $(\{1,2\},\{3,4,5\})$ but does not have any core-stable partition because agent(s) in the winning coalition can always deviate. So we found an example that has NTE and empty core.

We are going to characterize below the set of rules that has a core-stable partition for equal sharing.

Proposition 9. Let S^* be the minimally winning coalition of minimal size. Then under the equal sharing rule, the partition $\Pi = \{S^*, N-S^*\}$ is in the core if and only if $\sum_{i\in S^*-\{j\}} \pi_i < \sum_{i\in N-S^*} \pi_i$ for some $j \in S^*$.

Proof. ($\Leftarrow$) Let S^* be a winning coalition of the minimal size. Since $\sum_{i\in S^*-\{j\}} \pi_i < \sum_{i\in N-S^*} \pi_i$ for all $j \in S^*$, all agents in the winning coalition S^* are better off in S^*, it follows that $\{S^*, N-S^*\}$ is in the core by definition.
($\Rightarrow$) Suppose that $\Pi = \{S_1, S_2, ..., S_m\}$ is a core-stable partition where S_1 has the largest power and is of the minimal size. Then each agent in S_1 will get $M/|S_1|$ for which they are better off. Now suppose for a contradiction that $\sum_{i\in S_1-\{j\}} \pi_i > \sum_{i\in N-S_1} \pi_i$. Then the coalition $S_1 - \{j\}$ can deviate to $(S_1-\{j\}, \{j\}, S_2, ..., S_m)$ to get $M/|S_1-\{j\}|$ because $\sum_{i\in S_1-\{j\}} \pi_i > \sum_{i\in S_j} \pi_i$ for all $j = 2, 3, ..., m$. But this contradicts the fact that each agent in S_1 is better off with $M/|S_1|$. Hence if the core is nonempty, then we have $\sum_{i\in S^*-\{j\}} \pi_i < \sum_{i\in N-S^*} \pi_i$ for some $j \in S^*$ □

We have found some conditions of tournament games to have a core-stable partition for equal sharing case. Next, I would like to observe the conditions of tournament games to have NTE-stable partition.

Proposition 10. Under the equal sharing rule, a partition that has minimally winning coalition of minimal size is in NTE. In particular, it is nonempty.

Proof. Let S^* be the minimally winning coalition of the minimal size. Then it is clear that for any $l \in S^*$ we have $(\sum_{i\in S^*} \pi_i) - \pi_l < (\sum_{i\in N-S^*} \pi_i) + \pi_l$ because $\sum_{j\in S^*} \pi_j > 1/2 > \sum_{j\in S^*-k} \pi_j$ for all $k \in S^*$. Hence for equal sharing, NTE always exists and is the minimally winning coalition of minimal size. □

3.2. **Proportionally shared case.** Next, we look at the proportionally shared case. The proportionally shared case is slightly more complicated than equally shared case. It does not only depend on the power structure of the whole set N, but also the power structure of the coalition.

Proposition 11. For proportionally shared case, the core is nonempty if and only if there exists S^* such that $\sum_{i \in S^* - \{j\}} \pi_i < \sum_{i \in N - S^*} \pi_i$ for all $j \in S^*$.

Proof. ($\Leftarrow$) The same proof as equally shared case works. Let S^* be the winning coalition of the minimal weight. Since $\sum_{i \in S^* - \{j\}} \pi_i < \sum_{i \in N - S^*} \pi_i$ for all $j \in S^*$, all agents in the winning coalition S^* are better off in S^*, it follows that $\{S^*, N - S^*\}$ is in the core by definition. ($\Rightarrow$) Suppose that $\Pi = \{S_1, S_2, ..., S_m\}$ is a core-stable partition where S_1 has the largest power and is of the minimal s. Then each agent j in S_1 will get $\frac{\pi_j}{\sum_{i \in S_1} \pi_i} M$ for which they are better off. Now suppose for a contradiction that $\sum_{i \in S_1 - \{k\}} \pi_i > \sum_{i \in N - S_1} \pi_i$ for some $k \in S_1$. Then the coalition $S_1 - \{k\}$ can deviate to $(S_1 - \{k\}, \{k\}, S_2, ..., S_m)$ to get $\frac{\pi_j}{\sum_{i \in S_1 - \{k\}} \pi_i} M$ because $\sum_{i \in S_1 - \{k\}} \pi_i > \sum_{i \in S_l} \pi_i$ for all $l = 2, 3, ..., m$. But this contradicts the fact that each agent in S_1 is better off with $\frac{\pi_j}{\sum_{i \in S_1} \pi_i} M$. Hence if the core is nonempty, then we have $\sum_{i \in S^* - \{j\}} \pi_i < \sum_{i \in N - S^*} \pi_i$ for some $j \in S^*$ □

Proposition 12. For proportionally shared case, NTE always exists and equal minimally winning coalition of minimal weight.

Proof. Let S^* be the minimally winning coalition of minimal weight. Then we must have $\sum_{i \in S^* - \{j\}} \pi_i < \sum_{i \in N - S^* \cup \{j\}} \pi_i$ because we assume that there is no tie. But this implies that S^* is NTE by definition. □

4. With externality

Now, we consider the cases with externality. That means each agent cares not only about his own share, but also possibly cares about the other agents' shares. This relationship is represented by an $n \times n$ matrix for n agents. Let's look at some examples of the externality matrix to see what kind of power structures makes the core empty, but NTE exists.

Example 13. Consider $N = \{1, 2, 3, 4\}$ with 1 and 2 are Muslims, and 3 and 4 are Catholic. Agent 1 and 2 care about each other, but do not care about 3 and 4. Similarly, agent 3 and 4 care about each other, but do not care about 1 and 2. In this case, the externality matrix would be following:

$$M = \begin{bmatrix} 1 & \alpha & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & \beta \\ 0 & 0 & \beta & 1 \end{bmatrix}$$

with $1 > \alpha > 0$ and $1 > \beta > 0$.

4.1. **Equally shared case.** Observe that for what π the core is nonempty. Since agent 1 and 2 prefer to form a coalition and so do agent 3 and 4, Then for equal sharing, the final benefit will be the following.
If {1,2} are the coalition with the largest power, then we have

$$U_1(\frac{1}{2},\frac{1}{2},0,0) = \frac{1}{2}(1+\alpha)$$

$$U_2(\frac{1}{2},\frac{1}{2},0,0) = \frac{1}{2}(1+\alpha)$$

If {3,4} are the coalition with the largest power, then we have

$$U_3(0,0,\frac{1}{2},\frac{1}{2}) = \frac{1}{2}(1+\beta)$$

$$U_4(0,0,\frac{1}{2},\frac{1}{2}) = \frac{1}{2}(1+\beta)$$

Let $\pi = (.40, .22, .28, .10)$. Observe some partitions. It is not hard to see that the only possible partition to have core-stable partition and NTE is $(\{1,2\},\{3,4\})$. For this coalition, agent 1 prefer to be singleton, and can deviate because $\pi_1 > \pi_3 + \pi_4$. So for this π this game has NTE, but the core is empty.

4.2. **Proportionally shared case.** Let $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ and observe the payoff.

$$U_1 = x_1 + \alpha x_2 \qquad U_2 = \alpha x_1 + x_2$$

$$U_3 = x_3 + \beta x_4 \qquad U_4 = \beta x_3 + x_4$$

In this case, x_i depends on π, α, and β. If $\pi = (.40, .39, .11, .10)$, $\alpha = 1/10$, and $\beta = 1/10$, then agent 1 might prefer to form a coalition with agent 3 instead of agent 2 since
$v_1(\{1,2\},\{3,4\}) = \frac{40}{40+39} + \frac{1}{10}\frac{39}{40+39}$, and $v_1(\{1,3\},\{2,4\}) = \frac{40}{40+11}$, so $v_1(\{1,2\},\{3,4\}) < v_1(\{1,3\},\{2,4\})$.

Example 14. Consider $N = \{1,2,3,4\}$ with agent 1 and 2 are Yankees fans, and agent 3 and 4 are Mets fans. Agent 1 and 2 do not care about each other, but harm agent 3 and 4. Similarly, agent 3 and 4 do not care about each other, but hate agent 1 and 2. In this case, the externality matrix would be following,

$$M = \begin{bmatrix} 1 & 0 & \beta & \beta \\ 0 & 1 & \beta & \beta \\ \alpha & \alpha & 1 & 0 \\ \alpha & \alpha & 0 & 1 \end{bmatrix}$$

with $\alpha < 0$ and $\beta < 0$.

4.3. **Equally shared case.** Similarly to the previous case, agent 1 and 2 prefer to form a coalition and so do agent 3 and 4, Then for equal sharing, the final benefit will be the following.

If {1,2} are the coalition with the largest power, then we have

$$U_1(\frac{1}{2}, \frac{1}{2}, 0, 0) = \frac{1}{2}$$

$$U_2(\frac{1}{2}, \frac{1}{2}, 0, 0) = \frac{1}{2}$$

If {3,4} are the coalition with the largest power, then we have

$$U_3(0, 0, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$$

$$U_4(0, 0, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$$

Let $\pi = (.40, .22, .28, .10)$ as before. Observe some partitions. It is not hard to see that the only possible coalitions to have core and NTE is $(\{1, 2\}, \{3, 4\})$. For this coalition, agent 1 prefer to be singleton, and can deviate because $\pi_1 > \pi_3 + \pi_4$. So for this π this game has NTE, but the core is empty.

4.4. **Proportionally shared case.** Let $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ and observe the payoff.

$$U_1 = x_1 + \beta x_3 + \beta x_4 \qquad U_2 = x_2 + \beta x_3 + \beta x_4$$

$$U_3 = \alpha x_1 + \alpha x_2 + x_3 \qquad U_4 = \alpha x_1 + \alpha x_2 + x_4$$

As in the previous example, x_i depends on π, α, and β. But since $\beta < 0$, agent 1 can hardly form a coalition with agent 3. If we let $\pi = (.40, .39, .11, .10)$, $\alpha = 1/10$, and $\beta = -1/100$, then agent 1 might prefer to form a coalition with agent 3 instead of agent 2.

Example 15. Let's look at one more example, which is $N = \{1, 2, 3, 4\}$ with agent 1 and 2 are Muslim, and agent 3 and 4 are Jewish. Agent 1 and 2 care about each other, but harm agent 3 and 4. Similarly, agent 3 and 4 care about each other, but harm agent 1 and 2. In this case, the externality matrix would be following,

$$M = \begin{bmatrix} 1 & \alpha & \beta & \beta \\ \alpha & 1 & \beta & \beta \\ \gamma & \gamma & 1 & \delta \\ \gamma & \gamma & \delta & 1 \end{bmatrix}$$

with $1 > \alpha > 0$, $\beta < 0$, $\gamma < 0$, and $1 > \delta > 0$.

4.5. **Equally shared case.** Observe for what π is core nonempty. In this case agent 1 and 2 strongly prefer to form a coalition and so do agent 3 and 4, The final benefit will be the following.
If {1,2} are the coalition with the largest power, then we have

$$U_1 = (\frac{1}{2}, \frac{1}{2}, 0, 0) = \frac{1}{2}(1+\alpha)$$

$$U_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0) = \frac{1}{2}(1+\alpha)$$

If {3,4} are the coalition with the largest power, then we have

$$U_3 = (0, 0, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2}(1+\delta)$$

$$U_4 = (0, 0, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2}(1+\delta)$$

4.6. **Proportionally shared case.** Let $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ and observe the payoff.

$$U_1 = x_1 + \alpha x_2 + \beta x_3 + \beta x_4 \qquad U_2 = \alpha x_1 + x_2 + \beta x_3 + \beta x_4$$

$$U_3 = \gamma x_1 + \gamma x_2 + x_3 + \delta x_4 \qquad U_4 = \gamma x_1 + \gamma x_2 + \delta x_3 + x_4$$

In this case it is very hard for agent 1 to form a coalition with agent 3 and 4 because they are going to harm agent 1. But if we let
$\pi = (.40, .39, .05, .16)$, $\alpha = 1/10$, and $\beta = -1/10$, then agent 1 might prefer to form a coalition with agent 3 instead of agent 2 since
$v_1(\{1,2\},\{3,4\}) = \frac{40}{40+39} + \frac{1}{10}\frac{39}{40+39}$ and
$v_1(\{1,3\},\{2,4\}) = \frac{40}{40+5} - \frac{1}{10}\frac{5}{40+5}$, so
$v_1(\{1,2\},\{3,4\}) < v_1(\{1,3\},\{2,4\})$.

We have seen several examples to see how externality works in tournament games. Now, let's look at the specific example which has no NTE and empty core.

Example 16. We consider the model with following externality matrix for equally shared case;

$$M = \begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} \\ \frac{1}{3} & 0 & 1 \end{bmatrix}$$

and $\pi = (.35, .34, .31)$.
Suppose agent 1 and 2 form a coalition. Then the net utility is going to be

$$U_1(\frac{1}{2}, \frac{1}{2}, 0) = 1 \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} + 0 \times 0 = \frac{1}{2} + \frac{1}{6}$$

$$U_2(\frac{1}{2},\frac{1}{2},0) = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} + \frac{1}{3} \times 0 = \frac{1}{2}$$

$$U_3(\frac{1}{2},\frac{1}{2},0) = \frac{1}{3} \times \frac{1}{2} + 0 \times \frac{1}{2} + 1 \times 0 = \frac{1}{6}$$

In this case, agent 2 can deviate and form a new coalition with agent 3 to get more benefit.

$$U_1(0,\frac{1}{2},\frac{1}{2}) = 1 \times 0 + \frac{1}{3} \times \frac{1}{2} + 0 \times \frac{1}{2} = \frac{1}{6}$$

$$U_2(0,\frac{1}{2},\frac{1}{2}) = 0 \times 0 + 1 \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{2} + \frac{1}{6}$$

$$U_3(0,\frac{1}{2},\frac{1}{2}) = \frac{1}{3} \times 0 + 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$$

In this case, agent 3 can deviate and form a new coalition with 1 to get more benefit.

$$U_1(\frac{1}{2},0,\frac{1}{2}) = 1 \times \frac{1}{2} + \frac{1}{3} \times 0 + 0 \times \frac{1}{2} = \frac{1}{2}$$

$$U_2(\frac{1}{2},0,\frac{1}{2}) = 0 \times \frac{1}{2} + 1 \times 0 + \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

$$U_3(\frac{1}{2},0,\frac{1}{2}) = \frac{1}{3} \times \frac{1}{2} + 0 \times 0 + 1 \times \frac{1}{2} = \frac{1}{2} + \frac{1}{6}$$

In this case, agent 1 can deviate and form a new coalition with agent 2 to get more benefit. Therefore, this particular example has no NTE and empty core.

Above, we looked at simple externality matrices. Next I would like to observe a matrix which is more complicated.

Example 17. Let $\pi = (.35, .34, .31)$. Consider the following externality matrix,

$$M = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{6} \\ X & 1 & Y \\ \frac{1}{10} & \frac{1}{8} & 1 \end{bmatrix}$$

Then their final utilities are the followings.

$$U_1(x_1,x_2,x_3) = 1 \times x_1 + \frac{1}{3} \times x_2 + \frac{1}{6} \times x_3$$

$$U_2(x_1,x_2,x_3) = X \times x_1 + 1 \times x_2 + Y \times x_3$$

$$U_3(x_1,x_2,x_3) = \frac{1}{10} \times x_1 + \frac{1}{8} \times x_2 + 1 \times x_3$$

(1) If we let $X < Y$, then this game would have a cycle as in previous example because agent 1 prefer to form a coalition with agent 3, but agent 3 prefer to form a coalition with agent 2, but agent 2 prefer to form a coalition with agent 1, and so on. So this game has no NTE and empty core.
(2) If we let $X > Y$, then we would not make a cycle because agent 1 prefer to form a coalition with agent 2, and agent 2 prefer to form a coalition with agent 1. So $(\{1,2\},3)$ is NTE.

Lemma 18. *Let $S \subseteq N$. Assume that whenever $i \in S \subseteq N$ and $j \notin S$, we have $M_{ij} < \sum_{k\in S} M_{ik}/|S| < 1$. Then*
(1) *as we increase the size of coalition, any winning agent is worse off (cross-monotonicity), and*
(2) *agents who are not winning prefer to be winning.*

Proof. (1) Let $(\pi,\ M)$ be a game with $N = \{1,2,...,n\}$. Suppose we have a coalition {1,2}. If they are winning, then the utilities of the winning agents are $U_1(\frac{1}{2},\frac{1}{2},0...,0) = \frac{1}{2}(M_{11}+M_{12})$ and $U_2(\frac{1}{2},\frac{1}{2},0...,0) = \frac{1}{2}(M_{21}+M_{22})$. If agent 3 was added to this coalition, then the utilities of the winning agents would be $U_1(\frac{1}{3},\frac{1}{3},\frac{1}{3},0,...,0) = \frac{1}{3}(M_{11}+M_{12}+M_{13})$, and so on. We need $\frac{1}{2}(M_{11}+M_{12}) > \frac{1}{3}(M_{11}+M_{12}+M_{13}) > \frac{1}{4}(M_{11}+M_{12}+M_{13}+M_{14}) > ... > \frac{1}{n}(M_{11}+M_{12}+...+M_{1n})$ in order to say that any winning agent is worse off as we increase the size of a coalition. But if we know that $M_{ij} < \sum_{k\in S} M_{ik}/|S| < 1$, then we have $\frac{1}{k}(M_{11}+M_{12}+...+M_{1k}) > \frac{1}{k+1}(M_{11}+M_{12}+...+M_{1(k+1)})$ since $M_{1(k+1)} < \frac{1}{k}(M_{11}+M_{12}+...+M_{1k})$. Hence if we have $M_{ij} < \sum_{k\in S} M_{ik}/|S| < 1$, then as we increase the size of coalition, any winning agent is worse off.

(2) As in (1), let $(\pi,\ M)$ be a game with $N = \{1,2,...,n\}$, and suppose we have a coalition $S = \{1,2\}$. If they are winning, we know that they gets $1/2(M_{11}+M_{12})$ and $1/2(M_{21}+M_{22})$, and any other agent (losing) j gets $1/2(M_{j1}+M_{j2})$. But since we know that $M_{ij} < \sum_{k\in S} M_{ik}/|S| < 1$ for $i \in S \subseteq N$ and $j \notin S$, it follows that $1/2(M_{j1}+M_{j2}) < \sum_{k\in S\cup\{j\}} M_{jk}/|\{1,2,j\}|$. Hence if we have $M_{ij} < \sum_{k\in S} M_{ik}/|S| < 1$, then agents who are not winning prefer to be winning.

□

We have seen several examples which has empty core and which has nonempty core. Now we are interested in finding a set of externality matrices for which we can guarantee that the core is empty or nonempty for any π.

Theorem 19. *For equal sharing, we have the following results.*

(1) *If $n = 3$, given ε with $|\varepsilon| < 1/2$, then any matrix of the form*
$$M = \begin{bmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon \\ \varepsilon & \varepsilon & 1 \end{bmatrix}$$
always have a nonempty core for any π.

(2) *If $n = 3$, given ε with $|\varepsilon| < 1/2$, for any matrix of the type*
$$M \neq \begin{bmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon \\ \varepsilon & \varepsilon & 1 \end{bmatrix}$$
we can find π such that (π, M) has empty core.

(3) *For $n = 4$, for any matrix M there exists π that has empty core and $\tilde{\pi}$ that has a nonempty core.*

(4) *For $n \geq 5$, for any matrix M there exists π that has empty core and $\tilde{\pi}$ that has a nonempty core.*

Proof. (1) Let $\pi = \{\pi_1, \pi_2, \pi_3\}$ with $\pi_1 > \pi_2 > \pi_3$. If we have $\pi_1 > \frac{1}{2}$, then we are done because agent 1 can be singleton. Next, suppose that we have $\pi_1 < \frac{1}{2}$. Then the only possible partitions to have the core are the ones with minimally winning coalition, which are $(\{1,2\},\{3\})$, $(\{1,3\},\{2\})$, and $(\{2,3\},\{1\})$. But for the first two, agent 1 can deviate to get more benefit, so not in the core. Therefore, $(\{2,3\},\{1\})$ is the only possible partition to have the core. We need to analyze that $(\{2,3\},\{1\})$ is in the core for this game.
For this externality matrix, we get $U_1(x_1, x_2, x_3) = x_1 + \varepsilon x_2 + \varepsilon x_3$, $U_2(x_1, x_2, x_3) = \varepsilon x_1 + x_2 + \varepsilon x_3$, $U_3(x_1, x_2, x_3) = \varepsilon x_1 + \varepsilon x_2 + x_3$. Therefore, it does not matter with which agent to form a coalition, their share will be the same. It follows that this game can be now treated as if this is no-externality case except for the final benefit. Then it is clear that $(\{2,3\},\{1\})$ is in the core. Hence any matrix of this form has a nonempty core.

(2) Let $\pi = \{\pi_1, \pi_2, \pi_3\}$ and
$$M = \begin{bmatrix} 1 & M_{12} & M_{13} \\ M_{21} & 1 & M_{23} \\ M_{31} & M_{32} & 1 \end{bmatrix}$$
be externality matrix, not all M_{ij}'s are the same. Then we get $U_1(x_1, x_2, x_3) = x_1 + M_{12}x_2 + M_{13}x_3$, $U_2(x_1, x_2, x_3) = M_{21}x_1 + x_2 + M_{23}x_3$, $U_3(x_1, x_2, x_3) = M_{31}x_1 + M_{32}x_2 + x_3$. First of all, we have seen that if we have a cycle in the externality

matrix and $\pi_i < 1/2$ for $i = 1, 2, 3$, then clearly we have an empty core. There are two possibilities to have cycle, the one is $M_{12} > M_{13}, M_{23} > M_{21}, M_{31} > M_{32}$, and the other is $M_{12} < M_{13}, M_{23} < M_{21}, M_{31} < M_{32}$. Let's look at M_{ij}'s which do not make cycles.
Case 1: Suppose $M_{12} > M_{13}$. Then if $M_{21} > M_{23}$, then $(\{1,2\},\{3\})$ will be the winning coalition and if we let $\pi_1 > \pi_2 > \pi_3$, the game has an empty core. If $M_{21} < M_{23}$, then we have $M_{31} < M_{32}$ in order not to have a cycle. Then $(\{2,3\},\{1\})$ will be the winning coalition and if we let $\pi_2 > \pi_3 > \pi_1$, the game has an empty core.
Case 2: Suppose $M_{12} < M_{13}$. Then if $M_{31} > M_{32}$, then $(\{1,3\},\{2\})$ will be the winning coalition and if we let $\pi_1 > \pi_3 > \pi_2$, the game has an empty core. If $M_{31} < M_{32}$, then we have $M_{21} < M_{23}$ in order not to have a cycle. Then $(\{2,3\},\{1\})$ will be the winning coalition and if we let $\pi_2 > \pi_3 > \pi_1$, the game has an empty core. Hence for any matrix of this form, we can find π such that (π, M) has empty core.

(3) Let

$$M = \begin{bmatrix} 1 & M_{12} & M_{13} & M_{14} \\ M_{21} & 1 & M_{23} & M_{24} \\ M_{31} & M_{32} & 1 & M_{34} \\ M_{41} & M_{42} & M_{43} & 1 \end{bmatrix}$$

be the externality matrix. First of all, if we have $\pi_i > 1/2$ for some $i \in \{1,2,3,4\}$, then clearly the core is always nonempty since agent i can be singleton. So we only need to consider π's that make the core empty. Suppose that $\pi_i < 1/2$ for all $i \in \{1,2,3,4\}$ First, consider the matrix whose entries are all ε except for 1's on diagonal. Then if we let $\pi = (.32, .26, .19, .23)$, then the game has an empty core. Next, we analyze any matrix. As we have seen in (2), if M have a cycle, then the game has an empty core. Also, we know that if the externality matrix does not have a cycle, then at least two agents prefer to form a coalition with each other, say agent i and j. Then if we let $\pi_i > \pi_j > \pi_k > \pi_l$, then the game has an empty core.

(4) Let

$$M = \begin{bmatrix} 1 & M_{12} & .. & M_{1n} \\ M_{21} & 1 & .. & : \\ : & : & 1 & : \\ M_{n1} & .. & .. & 1 \end{bmatrix}$$

be the externality matrix. First of all, if we have $\pi_i > 1/2$ for some $i \in \{1, 2, ..., n\}$, then clearly the core is always nonempty. Therefore, we proved that for any matrix M there exists $\tilde{\pi}$ that has a nonempty core. Now we only need to consider π's that make the core empty. First, consider the matrix whose entries are all ε except for 1's on diagonal. As in (1), this game can be treated as if it is no-externality case. So if we let $\pi = (.36, .34, .12, .10, .08)$, for which we understood that the core is empty, then the game has an empty core. Next, we observe the matrix whose entries are not all the same. We know that if the matrix has a cycle, then the core is always empty. So suppose that M does not have a cycle. Then as before, at least two agents prefer to form a coalition with each other, say agent i and j. So if we let $\pi_i > \pi_j > \pi_k > ...$, then the game has an empty core. Hence For $n \geq 5$, for any matrix M there exists π that has empty core and $\tilde{\pi}$ that has a nonempty core.

□

5. Conclusions and future work

We have been analyzed two sharing rules to analyze agents who are looking to maximize their share of a divisible resource by being a singleton or forming coalitions. I do not believe that this situation is yet immediately applied to our real life. However, if we keep analyzing a lot of situations and use different types of sharing rules, I can assure you that it will become more useful tools to get more benefit. Therefore, I am looking to keep researching and hopefully find good strategies which can be applied in a lot of disciplines.

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