COMPLEXITY OF INDEX SETS OF COMPUTABLE LATTICES

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ABSTRACT

We analyze computable algebras in the sense of universal algebra and the index set complexity of properties of such algebras. We look at the difficulty of determining properties of $\text{Con}(A)$, the congruence lattice of an algebra $A$. In particular, we introduce formally the notion of a class of algebras witnessing the complexity of a property of algebras and show that computable lattices witness the $\Pi^0_2$-completeness of a computable algebras being simple, as well as witnessing the $\Sigma^0_3$-completeness of a computable algebras having finitely many congruences. Finally, in our main result, we show that the property “to be subdirectly irreducible” is $\Sigma^0_3$-complete as well, and in the process show that computable lattices witness this.
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CHAPTER 1
INTRODUCTION

When an undergraduate math student takes an algebra course, they may begin with the study of rings, groups, or perhaps monoids. Many computability researchers study computable rings or computable groups. It is interesting to consider the study of computable algebras in general, in the sense of universal algebra. In this paper we study properties of the congruence lattice of computable universal algebras and their complexity. In particular we show computable lattices witness $\Pi_2^0$-completeness of being simple, as well witnessing the $\Sigma_3^0$-completeness of having finitely many congruences. We also examine the complexity of the property “to be subdirectly irreducible”, which we determine to be $\Sigma_3^0$-complete.

The study of the complexity of being subdirectly irreducible has been studied by others, particularly in the finite case. In 1997, Ralph Freese showed one could decide if a finite computable lattice is subdirectly irreducible in time $O(n^2)$ where $n$ is the cardinality of the lattice in question. In 2002, Bergman et al. showed that for finite computable algebras the property “to be subdirectly irreducible” is NL-complete[3] also showing that directed graphs witness this complexity. We free ourselves of the finite restriction, and examine the complexity of these properties for possibly infinite algebras.

1.1 Definitions and Background

For the purpose of completeness, we state the definition of computable function.

Definition 1.1.1 (computable function). The class of computable functions $C$ is the smallest class of functions $C$ mapping $\omega$ to $\omega$
(i) containing functions

\[ \mathcal{O}(x) = 0, \mathcal{S}(x) = x + 1, \text{ and } \pi_i(\vec{x}) = x_i, \]

(ii) closed under composition,

(iii) closed under primitive recursion, that is if \( g, h \in \mathcal{C} \), the function

\[
f(\vec{x}, n) = \begin{cases} 
g(\vec{x}) & \text{if } n = 0 \\
h(\vec{x}, n - 1, f(\vec{x}, n - 1)) & \text{otherwise} \end{cases}
\]

is in \( \mathcal{C} \).

(iv) closed under \( \mu \)-recursion, that is given \( g \in \mathcal{C} \), where \( (\forall \vec{x})(\exists y)(g(\vec{x}, y) = 0) \), the function

\[
f(\vec{x}) = \text{the least } y \text{ where } g(\vec{x}, y) = 0
\]

is in \( \mathcal{C} \).

From now on however, rather than using the formal definition of computable function, we will freely appeal to Church’s Thesis, which asserts that a function is computable if and only if it is computable in the intuitive sense. To the reader to whom “intuitive sense” is inapplicable, it should suffice to know that if a function could be implemented in your favorite programming language, it is computable.

**Definition 1.1.2.** A sequence of functions \( \{f_i\} \) is **uniformly computable** if there exists a computable function \( f \) where \( f(i, \vec{x}) = f_i(\vec{x}) \) for all \( \vec{x} \). A **computable set** \( A \) is a subset of \( \omega \) whose characteristic function is computable. A **computable operation** on a computable set \( A \) is a computable function from \( A^k \) to \( A \) for some \( k \in \omega \).
Definition 1.1.3 (computable algebra). A computable algebra $A$ is a tuple $(A, F = (f_0, f_1, \ldots))$, where

(i) $A \subseteq \omega$ is computable and $A \neq \emptyset$,

(ii) $F$ is a sequence of uniformly computable operations,

(iii) there exists a computable function $\sigma : \omega \to \omega \cup \{\uparrow\}$, often called the arity function, where for any $k \in \omega$, if $\sigma(k) \in \omega$, $f_k$ is a computable operation from $A^{\sigma(k)}$ to $A$.

We call $A$ and $F$ the domain and operations of the algebra $A$ respectively. The value $\sigma(k)$ is the rank of $f_k$ or $\text{Rank } f_k$. If $\sigma(k) = \uparrow$, we say operation $f_k$ does not exist.

If $A$ is a computable algebra, we denote the domain as $\text{Dom } A$ and the operations as $\text{Op } A$. Often, as convention, we will assume implicitly that $\text{Dom } A = A$, and $\text{Op } A = (f_0, f_1, \ldots)$. Given an algebra, we define a computable subalgebra of $A$ to be a computable subset of $A$ closed under $\text{Op } A$. While it would be tempting to require $A$ to simply be an initial segment of $\omega$, by requiring $A$ only to be a computable subset of $\omega$, we allow for the definition of subalgebra to be the intuitive one.

Definition 1.1.4 (uniformly computable algebras). A sequence of algebras $\{A_i\}$ is uniformly computable if their characteristic functions $\{A_i\}$, arity functions $\{\sigma_i\}$, and operations $\{F_i\}$ are all uniformly computable.

A computable algebra is an effective version of a general algebra. Most structures studied in an upper division undergraduate algebra course are algebras in the general sense, such as groups, monoids, vector spaces, rings, fields, modules, etc. The study of properties of specific classes of computable algebras is an active area of computability research. We will focus on one particular class of algebras quite a bit, the class of lattices.
Definition 1.1.5 (computable lattice). A computable algebra \( L = (L, F) \) is a computable lattice if

(i) \( L \) has two binary operations, that is \( \sigma(0) = 2 = \sigma(1) \), and \( \sigma(n) = \uparrow \) for all \( n > 1 \), which are reflexive, commutative, and associative, and

(ii) \( f_0(n, f_1(n, m)) = n = f_1(n, f_0(n, m)) \) for all \( n, m \in \omega \).

To align our notation with that of lattice theorists, we will denote one of the operations \( \land \) and the other \( \lor \) and use infix rather than prefix notation. Lattices also define a partial ordering, \( \leq_L \), in particular where \( a \leq_L b \) if \( a \lor b = b \). If \( a <_L b \), and \( c <_L b \Rightarrow c \not<_L a \), we say \( b \) covers \( a \) or \( a \) is covered by \( b \), and we write this \( a \prec b \). Furthermore for \( a, b \in L \), if \( a \leq_L b \), we define the interval \([a, b]_L\) as the set \( \{x \in L : a \leq_L x \leq_L b\} \). We will call subalgebras of a lattice sublattices, and note that an interval of a lattice is a sublattice. If \( A \) is a sublattice of \( L \), we say \( L \) is a superlattice of \( A \).

We will refer often to the following well known family of lattices.

Definition 1.1.6. For \( n \in \omega \) or \( n = \omega \), define \( M_n = (M_n, (f_1 := \land, f_2 := \lor)) \) as follows. Let \( M_n = \{0, 1, \ldots, n + 2\} \) if \( n \in \omega \) and \( M_n = \omega \) when \( n = \omega \). Let \( 0 <_L 1 \) and for all \( m > 1 \in M_n \) we have \( 0 \prec m \prec 1 \). This results in the lattice in Figure 1.1.

![Figure 1.1: The lattice \( M_n \)](image)

Definition 1.1.7 (substitution property). Let \( A = (A, (f_0, f_1, \ldots)) \) be a computable algebra and \( \theta \) a binary relation on \( A \). We use the shorthand \( a \theta b \) if \((a, b) \in \theta\).
We say $\theta$ has the substitution property if for every $n \in \omega$ where $\sigma(n) \neq \uparrow$, we have if
\[ a_1 \theta b_1, \quad a_2 \theta b_2, \quad \ldots, \quad a_{\sigma(n)} \theta b_{\sigma(n)} \]
then
\[ f_n(a_1, a_2, \ldots, a_{\sigma(n)}) \theta f_n(b_1, b_2, \ldots, b_{\sigma(n)}). \]

Intuitively, a relation has the substitution property if it “respects” the operations of an algebra. In the case where the relation is reflexive, symmetric, and transitive we distinguish it from other operations.

**Definition 1.1.8 (congruence).** If $A$ is a computable algebra, and $\theta$ a binary relation on $A$, we say $\theta$ is a congruence if $\theta$ is an equivalence relation which has the substitution property. The set of all congruences of $A$ we call $\text{Con}(A)$. For $a \in A$, we define the congruence class of $a$ under $\theta$ as the set $\{b \in A : a \theta b\}$ and denote this $[a]_\theta$. Occasionally, when the context is clear we will suppress the subscript and write only $[a]$.

Congruences of algebras correspond to homomorphic images of that algebra, that is, every congruence induces a homomorphism, namely the map from the algebra to its congruence class. Every algebra $A$ has congruences $0_A = \{(x, x) : x \in A\}$ and $1_A = A \times A$. If these two congruences are the same, $|A| = 1$ and we call $A$ trivial. If $\text{Con}(A) = \{0_A, 1_A\}$ and $A$ is not trivial; we call $A$ simple.

It is known for instance that the lattice $M_n$ is simple when $n \geq 3[?]$. We will use this fact heavily.
1.2 Index Sets of Properties of Computable Algebras

**Definition 1.2.1** (principal congruence). Given a computable algebra $A$, and element $a, b \in A$, we say

$$C_{gA} (a, b) = \bigcap \{ \theta \in \text{Con}(A) : (a, b) \in \theta \}.$$  

It is known that $C_{gA} (a, b)$ is a congruence on $A$[2, Prop 1.23] and is thus the smallest congruence $\theta$ where $a \theta b$. As before, when the context is clear, we will suppress the subscript, and just write $C_g (a, b)$.

**Definition 1.2.2** $(m$-reducibility). A set $A$ is many-one reducible to $B$ ($A \leq_m B$) if there exists a computable function $f$ such that for all $a, a \in A \Leftrightarrow f(a) \in B$.

We abbreviate many-one reducible as $m$-reducible. If $A \leq_m B$ and $B \leq_m A$ we say $A \equiv_m B$.

It is known [9, Theorem I] that $\leq_m$ is reflexive and transitive, and that $\equiv_m$ is an equivalence relation. The equivalence classes are called many-one degrees or $m$-degrees.

We use the main definition from Khoussainov and Morozov[?, Main Definition].

**Definition 1.2.3** ($J$-complete property of an algebra). Let $J \subseteq 2^\omega$ be closed downward under $m$-reducibility. We say that the property $B$ of algebras in a class $\mathcal{K}$ is $J$-complete, if the following is true:
(i) For all uniformly computable sequences of algebras $\{A_i\}_{i<\omega}$ in $\mathcal{K}$,

$$\{i : A_i \text{ satisfies } B\} \in J.$$

(ii) There exists a uniformly computable sequence of algebras $\{A_i\}_{i<\omega}$ in $\mathcal{K}$ for which for all $J \in \mathcal{J}$,

$$J \leq_m \{i : A_i \text{ satisfies } B\}.$$

If the class $\mathcal{K}$ is the class of all algebras, we suppress mention of it. If a property $B$ satisfies property (1) we say $B$ is a $J$-property (in $\mathcal{K}$). If it satisfies property (2) we say $B$ is $J$-hard (in $\mathcal{K}$).

We also introduce the notion of a class of algebras $\mathcal{S}$ witnessing $J$-completeness for a class $\mathcal{K}$. If a property $B$ is $J$-complete in $\mathcal{K}$, and if there exists a uniformly computable sequence of algebras in a subclass $\mathcal{S}$ of a class $\mathcal{K}$ witnessing that $B$ is $J$-hard in $\mathcal{K}$, we say that $\mathcal{S}$ witnesses the $J$-completeness of $B$ in $\mathcal{K}$. Intuitively, this means that knowing the solution to determine if any $s \in \mathcal{S}$ has property $B$ is sufficient to determine if any $k \in \mathcal{K}$ has property $B$, and as such, $\mathcal{S}$ is somehow sufficiently complex in property $B$ to understand the property in general.

Given a sequence of computable algebras $\{A_i\}$ and some subset $S$ of that sequence, the set $\{i : A_i \in S\}$ is called the index set of $S$. We shall examine complexities of index sets of universal algebras. To relate these complexities, we introduce a well known hierarchy of sets closed under $\leq_m[10, \text{Theorem 1.3(v)}]$.

**Definition 1.2.4** (The Arithmetical Hierarchy). The *arithmetical hierarchy* is a method of categorizing some sets according to their expressibility.
(i) A set $B$ is $\Sigma^0_n$ if there is a computable function $P(x, y_1, \ldots, y_n)$ such that

$$x \in B \iff (\exists y_1)(\forall y_2)(\exists y_3) \cdots (Q y_n)(P(x, y_1, \ldots, y_n) = 1),$$

where the quantifiers alternate and thus $Q = \exists$ if $n$ is odd and $Q = \forall$ if $n$ even.

(ii) A set $B$ is $\Pi^0_n$ if there is a computable relation $P(x, y_1, \ldots, y_n)$ such that

$$x \in B \iff (\forall y_1)(\exists y_2)(\forall y_3) \cdots (Q y_n)(P(x, y_1, \ldots, y_n) = 1),$$

where the quantifiers alternate and thus $Q = \forall$ if $n$ is odd and $Q = \exists$ if $n$ even.

Khoussainov and Morozov for instance, proved the following concerning properties of computable universal algebras:[?]

(1) The property “to be simple” is $\Pi^0_2$-complete,

(2) The property “to have finitely many congruences” is $\Sigma^0_3$-complete.

In proving statement (1) and (2), Khoussainov and Morozov proved that computable groups witness $\Pi^0_2$-completeness of being simple as well as witnessing $\Sigma^0_3$-completeness of having finitely many congruences. We prove that lattices are also sufficient, i.e. groups can be replaced by lattices, and thus also give alternate proofs for (1) and (2) as corollaries. In addition, we examine the complexity of the property “to be subdirectly irreducible” which we will define below.

In order to prove these results we will need a bit more theory.

**Definition 1.2.5** (basic unary operation). Given an algebra $A = (A, \{f_i\}_{i<\omega})$, and $i$ such that if $\sigma(i) = n$, then for any $a_0, \ldots, a_{n-1} \in A$, $j < n$, we have

$$u(x) = f_i(a_0, a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_{n-1})$$
is a basic unary operation.

The next result is one implicitly used and proved by Malcev, (see [7, Theorem 1.10.3] for a treatment of this).

**Theorem 1.2.6** (Malcev’s Lemma). *For an algebra* \( A \), \((a, b) \in Cg_A (c, d) \) *if and only if there exists* \( n < \omega \), *a sequence* \( a = z_0, z_1, \ldots, z_n = b \) *of elements of* \( A \) *and a sequence of basic unary operations* \( u_0, \ldots, u_{n-1} \) *such that*

\[
\{u_i(c), u_{i+1}(d)\} = \{z_i, z_{i+1}\}
\]

*for all* \( i < n \).

Further, we note that any element of \( \omega \) can be interpreted computably as a finite sequence. For instance given \( i = p_1^{a_1+1} \cdot p_2^{a_2+1} \cdot p_n^{a_n+1} \) where \( p_1 < p_2 < \ldots < p_n \) are primes, we can interpret \( i \) as the sequence \((a_1, a_2, \ldots, a_n)\). Conversely, any sequence of length \( n \) can be computably encoded as an \( i \) simply by using the first \( n \) primes and this encoding scheme.

Thus every basic unary operation \( u(x) = f_i(a_0, a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_{n-1}) \) of an algebra \( A \) can be encoded computably as an element of \( \omega \), by encoding the sequence

\[(i, j, a_0, a_1, \ldots, a_n).\]

Interpreting elements of \( \omega \) as sequences as above is a common trick, and we will now do so implicitly throughout the rest of this dissertation. With this in mind, we can now make sense of, state, and prove the following lemma which will be used in each of the main results to come.
Lemma 1.2.7. Given a computable algebra $A$ and $c, d \in A$, the set

$$\{(a, b) : (a, b) \in C_{g_A}(c, d)\}$$

is $\Sigma^0_1$.

Proof. We define a computable boolean function $R(a, b, c, d, x)$. Since $A$ is computable, we can check to see if $a, b, c, d \in A$. Given $x \in \omega$, we can computably determine if $x$ can be interpreted as an ordered pair, say $(\vec{z}, \vec{u})$, coded by some fixed convention. Furthermore we can interpret $\vec{z}$ as a finite sequence $(z_0, z_1, \ldots, z_n)$ and check to see if $z_0 = a$ and $z_n = b$. We interpret $\vec{u}$ as a sequence of unary operations, and check to see if they are valid, as well as the length of $\vec{u}$ is $n - 1$. Finally we check if $\{u_i(c), u_{i+1}(d)\} = \{z_i, z_{i+1}\}$ for $i < n$. If any of these checks fail we let $R(a, b, c, d, x) = 0$ otherwise $R(a, b, c, d, x) = 1$. Then

$$(a, b) \in C_{g_A}(c, d) \Leftrightarrow (\exists x)R(a, b, c, d, x).$$

So $\{(a, b) : (a, b) \in C_{g_A}(c, d)\}$ is $\Sigma^0_1$. \hfill \qed

We will also use another useful theorem, from [4, Lemma 2.1]

Theorem 1.2.8. If $L$ is a lattice, and $a \leq_L b$ then in $C_{g_L}(a, b)$ the following are equivalent:

(i) $[a] = [a, b]_L$ and for all $x \notin [a, b]_L$, $[x] = \{x\}$;

(ii) for all $x \in L$, if $x \notin [a, b]_L$ then $(x \leq_L b \Rightarrow x <_L a)$ and $(x \geq_L a \Rightarrow x >_L b)$.

We will implicitly use the following lemma as well.

Lemma 1.2.9. If $a \leq b \in L$, then for all $x \in [a, b]_L$, $(x, a) \in C_{g}(a, b)$
Proof. Since \((a,b) \in C_g(a,b)\) we have \((a,x) = (x \land a, x \land b) \in C_{gL}(a,b)\).

We will also need a technical lemma concerning simple lattices.

**Lemma 1.2.10.** Suppose \(M = [0_M, 1_M]_L, N = [0_N, 1_N]_L\) are simple sublattices of a lattice \(L\), \(\{1_M, 0_N\} \subseteq M \cap N, 1_M \neq 0_N\), and \([0_M, 1_N]_L = M \cup N\). Then \([0_M, 1_N]_L\) is simple.

Proof. Let \(a \neq b \in M \cup N\). If \(a,b \in M\), since \(M\) simple, for all \(x,y \in M\), we have \((x,y) \in C_{gL}(a,b)\). But then \((1_M, 0_N) \in C_{gL}(a,b)\), so since \(N\) is simple, we have for all \(x,y \in N\), \((x,y) \in C_{gL}(1_M, 0_N) \subseteq C_{gL}(a,b)\). So \(C_{gL}(a,b)1_{[0_M,1_N]}_L\). The same argument works with \(a,b \in N\). Thus we only need to consider the case where \(a \in M \setminus N\) and \(b \in N \setminus M\).

In that case \(a \lor 1_M = 1_M\), and \(b \lor 1_M \in N\), we would have two distinct elements of \(N\) related, so the above argument applies. Thus \([0_M, 1_N]_L\) is simple.
CHAPTER 2
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Theorem 2.0.11. The computable lattices witness $\Pi^0_2$-completeness of a computable algebra being simple.

Proof. We first show given a sequence of algebras $\{A_i\}$, that

$$\{i : A_i \text{ is simple} \} \in \Pi^0_2.$$ 

This is shown by noting that in a simple algebra $A_i$, if $a \neq b$, then $Cg(a, b) = 1_{A_i}$. Thus

$$\{i : A_i \text{ is simple} \} = \{i : (\forall a, b, c, d)(a, b, c, d \in A_i, c \neq d \rightarrow (a, b) \in Cg(c, d))\}$$

$$= \{i : (\forall a, b, c, d)(\exists x)(\neg(a, b, c \neq d \in A_i) \text{ or } R(a, b, c, d, x))\},$$

which is $\Pi^0_2$, where $R$ is as in Lemma 1.2.7.

Now suppose $B$ is $\Pi^0_2$. Then it has been shown[9, Lemma for 14.XV] that there exists a computable sequence of boolean function $P_b(t)$ such that

$$b \in B \iff (\exists t)(P_b(t) = 1),$$

where $(\exists t)$ means “there exists infinitely many $t$”. We construct a sequence of lattices $\{L_b\}_{b<\omega}$. We define each in stages, and will also define a helper function $T(s)$ which will, informally, be the “top” of the lattice as currently defined by stage $s$.

Stage 0: We start every $L_b$ with the lattice operations in Figure 2.1 We also let
Figure 2.1: The initial lattice

$T(0) = 3$, in keeping with our desire to have $T(0)$ point at the top after stage $t$. For technical reasons which will be clear later, we also let $T(-1) = 2$.

Stage $t$: We will define our lattice operations for all elements below $8 + 3t$. In particular, we just define our lattice operations for additional elements $6 + 3t, 7 + 3t$ and $8 + 3t$.

If $P_b(t) = 0$ we let $0 \prec 6 + 3t, 7 + 3t, 8 + 3t \prec 2$. and since we have not put anything above $T(t)$, we let $T(t) = T(t - 1)$.

Otherwise, if $P_b(t) = 1$, we let

$$T(t - 1) + 1 \prec 7 + 3t, 8 + 3t \prec T(t - 1) \prec 6 + 3t.$$
We also let $T(t) = 6 + 3t$. (See Figure 2.4)

![Figure 2.3: Placements of nodes when $P_b(t) = 1$](image)

We claim under this construction, that

$$B = \{ b : L_b \text{ is simple} \},$$

and thus $B \equiv_m \{ b : L_b \text{ is simple} \}$ under the identity function.

Suppose $b \not\in B$. Then there exists $s_0$ where for all $t \geq s_0$, $P(b, t) = 0$. Thus, for all $t \geq s_0$, $T(t) = T(s_0)$. Also, there exists some maximal $s_1 < s_0$ where $T(s_0) > T(s_1)^*$. By Theorem 1.2.8 we have that $0_{L_b} \neq \text{Cg}(T(s_0), T(s_1)) \neq 1_{L_b}$. Thus $L_b$ is not simple.

Suppose $b \in B$. Let $t_k$ = the $k$th $t$ such that $T(t) > T(t - 1)$. Since $b \in B$, this sequence is defined for all $k \in \omega$. Also let $t_0 = -1$. Then $T(t_k) \prec T(t_{k+1})$ for $k \in \omega$.

*This was the reason for letting $T(-1) = 2.$
Then \([0, T(t_0)]_L\) is isomorphic to \(M_n\) for some \(n < \omega + 1\), which is simple. Since for every \(k > 0\), we have \([T(t_k) + 1, T(t_k)]_L\) isomorphic to \(M_3\) by construction, also simple, then by Lemma 1.2.10, we have for any \(k\), \([0, T(t_k)]_L\) is a simple sublattice. We claim this shows \(L_b\) is simple, since for any \(a \neq b\), \(c \neq d\), there exists some \(t_k\) above all of them in the lattice theoretical sense, which gives \((c, d) \in Cg(a, b)\) inside of \([0, T(t_k)]_L\). But this gives a sequence required by Theorem 1.2.6 which will exist in the superlattice \(L_b\). Thus \(a \neq b\) forces \((c, d) \in Cg(a, b)\), which gives \(L_b\) is simple.

\[\text{Figure 2.4: } L_b \text{ when } b \in B\]

**Theorem 2.0.12.** The computable lattices witness \(\Sigma_3^0\)-completeness of a computable algebra having finitely many congruences.

**Proof.** We first show property (1). Suppose an algebra \(A\) has infinitely many congruences. Let \(\{\theta_i\}_{i<\omega} \in \text{Con}(A)\) where \(\theta_i \neq \theta_j\) for \(i \neq j\). We build a sequence of pairs recursively. Let \((a_0, b_0) \in \theta_0\). For every \(i > 1\), there exists \((a_i, b_i) \in A \times A\) where \((a_i, b_i) \in \theta_k\) for some \(k \leq i\) and \(Cg_A(a_i, b_i) \neq Cg_A(a_j, b_j)\) for all \(j < i\). Thus we have
infinitely many principal congruences. If an algebra has finitely many congruences, then surely it has finitely many principal ones. Thus an algebra has finitely many congruences if and only if it has finitely many principal ones. Thus an algebra $A$ has finitely many congruences if and only if

$$\exists x = (a_0, b_0, a_1, b_1, \ldots, a_k, b_k) \left( \forall a, b \left( \bigvee_{i \leq k} Cg(a, b) = Cg(a_i, b_i) \right) \right),$$

so “having finitely many congruences” is a $\Sigma^0_3$ property. Now we show property (2). Since $B$ is $\Sigma^0_3$, there is a uniformly computable sequence of relations $P_b(i, j)$ where

$$b \in B \iff (\exists i)(\exists j)(P_b(i, j) = 1)$$

[?, Theorem 14.XVII and 14.XV]rogers We construct a sequence of lattices where

$$B = \{ b : L_b \text{ has finitely many congruences} \},$$

which will show property (2) under the identity function. Let $L_b = \omega$. For the purposes of notation, let $i_k = 2(i + 4k)$ for $i < 5$. Note that $1_k = 0_{k+1}$ for all $k$. For uniformity we will write $1_{-1} = 0$. We let $0_k < 2_k, 3_k, 4_k < 1_k$ for all $k \in \omega$. This yields the lattice with relations in Figure 2.5. We build $L_b$ in stages. We first define a helper function $c_s(n)$. Let $c_0(n) = n$ for all $n \in \omega \cup \{-1\}$. Given $s$, let $i$ and $j$ be the numbers where $i \leq j$ and $s = \frac{i(j+1)}{2} + i$. We define the functions $\tilde{i}(s) = i$ and $\tilde{j}(s) = j$ to describe this relation for any $s$. We let

$$c_{s+1}(n) = \begin{cases} c_s(n) & \text{if } P_b(i, j) = 0 \text{ or } n < i, \\ c_s(n + 1) & \text{otherwise.} \end{cases}$$
If \( P_b(i, j) = 0 \) we let \( 0_1 \prec 2s+1, 2s+3 \prec 2_0 \). Otherwise, we let \( 2_{c_s(i)} \prec 2s+1, 2s+3 \prec 2_{c_s(i)+1} \).

We claim the \( \{L_b\} \) satisfy. For this we first prove a small claim

**Claim** (For all \( s \), \( [1_{c_s(n-1)}, 1_{c_s(n)}]_L \) is a simple sublattice of \( L_b \)). We proceed by induction.

At \( s = 0 \), \( [1_{c_s(n-1)}, 1_{c_s(n)}]_L \cong M_3 \) for all \( n \in \omega \) and are thus simple.
Now suppose \([1_{cs(n-1)}, 1_{cs(n)}]_L\) simple for all \(n \in \omega\). If \(P_b(i, j) = 0\), we have 
\([1_{cs+1(n)}, 1_{cs+1}(n+1)]_L = [1_{cs(n)}, 1_{cs(n+1)}]_L\) which are simple for all \(n \in \omega\), so suppose not.

Then \(c_s(n-1) = c_{s+1}(n-1)\) and \(c_s(n) = c_{s+1}(n)\) for \(n < \overline{i}(s)\), and for all \(n \geq \overline{i}(s)\), we have \(c_{s+1}(n) = c_s(n+1)\). So we need really only worry about \([1_{cs+1(i-1)}, 1_{cs+1(i)}]_L\). By construction \(c_{s+1}(i-1) = c_s(i-1)\), and \(c_{s+1}(i) = c_s(i+1)\). We know \([1_{cs(i-1)}, 1_{cs(i)}]_L\) and \([1_{cs(i)}, 1_{cs(i+1)}]_L\) are simple by induction hypothesis. But \([2_{cs(i)}, 2_{cs(i)+1}]_L \cong M_3\) which is simple, and so Lemma 1.2.10 applies and we get \([1_{cs(i-1)}, 2_{cs(i)+1}]_L\) is simple. We apply Lemma 1.2.10 once more and this gives \([1_{cs(i-1)}, 1_{cs(i+1)}]_L\) is simple. Thus \([1_{cs+1(i-1)}, 1_{cs+1(i)}]_L\) is simple.

End of Claim.

Suppose \(b \in B\). Then there exists \(i\) where there are infinitely many \(j\) such that 
\(P_b(i, j) = 1\). Let \(i_0\) be the smallest such \(i\). Since \(i_0\) is the smallest such \(i\), there exists a smallest \(j_0\) where for all \(j \geq j_0\), if \(i < i_0, P_b(i, j) = 0\). In particular that means

Figure 2.7: Placements of nodes when \(P_b(i, j) = 1\)
that at some point $s_0$, if $s \geq s_0$ we have $c_s(i_0 - 1) = c_{s_0}(i_0 - 1)$. We claim that $\{x : x \geq L \ 1_{c_{s_0}(i_0-1)}\}$ is a simple sublattice.

Let $a \neq b, c, d \geq L i_i$. Since every time $j \geq \max\{i_0, j(s)\}$ where $P_b(i_0, j) = 1$, we get $c_{s+1}(i_0) = c_s(i_0 + 1)$, we know that $c_s(i_0)$ is unbounded in $s$. Then there exists some $s > s_0$ where $a, b, c, d \leq L \ 1_{c_s(i)}$. But by the above claim $[1_{c_{s_0}(i_0 - 1)}, 1_{c_s(i_0)}]_L$ is simple, so $c, d \in C_g(a, b)$. This gives $\{x : x \geq L \ 1_{c_{s_0}(i_0-1)}\}$ simple. But $[0_0, 1_{c_{s_0}(i_0-1)}]_L$ is finite, so there can only be finitely many principal congruences with elements in $[0_0, 1_{c_{s_0}(i_0-1)}]_L$.

If $b \notin B$, we have infinitely many simple intervals connected at the top and bottom at a single node. By Theorem 1.2.8, each of these have independent congruences, and so there are at least $2^\omega$ congruences, which is infinite. \qed
CHAPTER 3
THE COMPLEXITY OF “TO BE SUBDIRECTLY IRREDUCIBLE”

Definition 3.0.13 (subdirectly irreducible). A computable algebra \( A \) is subdirectly irreducible there exist two elements \( a \neq b \in A \), where for any \( \theta \neq 0_A \in \text{Con}(A) \), we have \((a, b) \in \theta\). The congruence \( C_g(a, b) \) is called the monolith of \( A \).

This definition is equivalent to the one more commonly seen, that a universal algebra \( A \) is subdirectly irreducible when \(|A| > 1 \) and if \( A \) is isomorphic to a subgroup of a direct product of algebras, then \( A \) is isomorphic to a subgroup of one of the factors. The subdirect representation theorem\([?]\) of universal algebra states that every algebra is subdirectly representable by its subdirectly irreducible quotients.

Our goal is to show that the property “to be subdirectly irreducible” is \( \Sigma_3^0 \)-complete. To show this property to be \( \Sigma_3^0 \)-complete, for any \( \Sigma_3^0 \) set \( B \) we will construct a sequence of lattices \( \{L_b\} \) where \( \{L_b\} \) is subdirectly irreducible when \( b \in B \). Each of these will have as a sublattice the following sublattice \( \mathbb{L} = (2 \omega, \wedge|_L, \vee|_L) \).

For the purposes of notation, let \( i_k = 2(i + 4k) \) for \( i < 4 \). We will also refer to the sublattice \( R_k = (\{i_k\}_{i<4}, \wedge|_{R_k}, \vee|_{R_k}) \). For \( k \leq j \in \omega \), define \( \wedge \) and \( \vee \) on \( 2\omega \) with the following relations. For \( k \in \omega \), let \( 0_k < 2_k, 3_k < 1_k \). Furthermore, Let \( 0_k \prec 0_{k+1} \) and \( 1_{k+1} \prec 1_k \). This yields the sublattice in Figure 3.

We first show a few properties of this sublattice.

Lemma 3.0.14. If \( i_k \neq x \in R_k \) then \( C_g(i_k, x) = C_g(0_k, 1_k) \) and for all \( c, d \notin [0_k, 1_k]_L \) where \( c \neq d, (c, d) \notin C_g(0_k, 1_k) \).

That is, the congruence classes of \( C_g(i_k, j_k) \) are \( \{0\}, \{1\}, \ldots \{0_k - 1\}, \{n \colon n \geq 0_k\} = [0_k, 1_k]_L \).
Proof. Consider the homomorphism with kernel $C g (0_{k+1}, 1_{k+1})$, say $\varphi$. By Theorem 1.2.8, the only congruence class with more than one element in it (nontrivial congruence class) is $[0_{k+1}] = [0_{k+1}, 1_{k+1}]_L$. Under $\varphi$, $[0_k, 1_k]_L \cong M_3$ which is simple. Thus Mal’cev’s Lemma gives us a series of basic unary operations and a se-
quence of elements of $\varphi(L)$ witnessing $(\varphi(0_k), \varphi(1_k)) \in \text{Cg}(i_k, x)$. The same unary operations will and sequence will work in $L$ in the obvious way, thus witnessing $(0_k, 1_k) \in \text{Cg}(i_k, x)$. Since $i_k, x \in [0_k, 1_k]_L$, we get $(i_k, x) \in [0_k, 1_k]_L$ by Lemma 1.2.9. Thus $\text{Cg}(i_k, x) = \text{Cg}(0_k, 1_k)$ in $L$.

The second part follows directly by nothing that $[0_k, 1_k]_L$ satisfies the conditions of Lemma 1.2.8.

\[\Box\]

**Corollary 3.0.15.** Every non-trivial $\theta \in \text{Con}(L)$ is principal. In particular each such $\theta$ is of the form $\text{Cg}(0_k, 1_k)$ for some $k \in \omega$.

**Proof.** Let $\theta \in \text{Con}(L)$ be non-trivial. Then there exists minimum $i_k < j_m$ where $i_k \theta j_m$ and $k, m < 4$. We claim $\theta = \text{Cg}(0_k, 1_k)$.

By Lemma 3.0.14, we have $\text{Cg}(i_k, j_m) = \text{Cg}(0_k, 1_k)$, so $\text{Cg}(0_k, 1_k) \subseteq \theta$. But any for $(x, y) \in \theta$ where $x, y \geq i_k$ we have $x, y \geq 0_k$, and so $x, y \in [0_k, 1_k]_L$, and thus by Lemma 1.2.9 we have $(x, y) \in \text{Cg}(0_k, 1_k)$. This gives $\theta \subseteq \text{Cg}(0_k, 1_k)$.

\[\Box\]

**Lemma 3.0.16.** $\text{Con}(L) \cong (\omega, (\land = \max(x, y), \lor = \min(x, y))) =: \omega^d$.

**Proof.** By Corollary 3.0.19 we have every element of $\text{Con}(L)$ is principal. Furthermore, by Corollary 3.0.14, we know they are the form $\text{Cg}(0_k, 1_k)$. Let $\varphi : \text{Con}(L) \to \omega^d$ by $\text{Cg}(0_k, 1_k) \mapsto k$. Since if $k \geq j$, $\text{Cg}(0_k, 1_k) \subseteq \text{Cg}(0_j, 1_j)$ we have $\text{Cg}(0_k, 1_k) \lor \text{Cg}(0_j, 1_j) = \text{Cg}(0_{\min(j, k)}, 1_{\min(j, k)})$ and similarly for $\land$.

\[\Box\]

We are now ready to prove our main result.

**Theorem 3.0.17.** The index set of all computable algebras which are subdirectly irreducible is $\Sigma^0_3$-complete.

**Proof.** We first show that given a computable sequence of algebras $\{A_i\}_{i \in \omega}$ the set $X := \{i : A_i \text{ is subdirectly irreducible}\}$ is a $\Sigma^0_3$ set. We have $A_i$ is subdirectly
irreducible if and only if:

$$(\exists a,b \forall c,d) (a \neq b, c \neq d \& a, b, c, d \in A_i \Rightarrow (a, b) \in \text{Cg}(c, d)).$$

Since $A_i$ is a computable algebra, $A_i$ is computable, and by Lemma 1.2.7 we have that the statement $a \neq b, c \neq d \& a, b, c, d \in A_i \Rightarrow (a, b) \in \text{Cg}(c, d)$ is $\Sigma^0_1$. Thus $X$ is $\Sigma^0_3$.

It only remains to show that $X$ is $\Sigma^0_3$-hard. Let $B$ be $\Sigma^0_3$. We are now ready to construct our sequence of lattices with the property that $L_b$ is subdirectly irreducible if and only if $b \in B$.

As in Theorem 2.0.12 we note that there is a uniformly computable sequence of relations $P_b(i, j)$ where

$$b \in B \Leftrightarrow (\exists i)(\exists j)(P_b(i, j) = 1).$$

We let $L_b = \omega$. We build $L_b$ in stages, starting with $L$ from above. We use the same $c_s(n)$ from Theorem 2.0.12 along with the $i, j$ notation. Informally, $c_s(n)$ will be the largest number such that $\text{Cg}(0_{c_s(n)}, 1_{c_s(n)})$ is the $n^{th}$ largest congruence of $L_{i,s},$
the lattice constructed by stopping after stage $s$.

At stage $s$ we define our lattice operations for $2s + 1$. If $P_b(i(s), j(s)) = 0$, we let $0_0 < 4s + 1 < 1_0$.

If $P_b(i(s), j(s)) = 1$, we let $2c_s(n+1) < 2s + 1 < 1_{c_s(n)}$.

This defines a lattice $L_b$. We make some claims concerning this lattice.

**Lemma 3.0.18.** For all $s, n$, if $c_s(n) \leq m < c_s(n + 1)$ then $C_g(0_{c_s(n)}, 1_{c_s(n)}) = C_g(0_m, 1_m)$. 

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Proof. This is clear from the construction. 

Lemma 3.0.19. Every non-trivial \( \theta \in \text{Con}(L) \) is principal. In particular each such \( \theta \) is of the form \( Cg(0_k,1_k) \) for some \( k \in \omega \).

Proof. Let \( \theta \in \text{Con}(L) \) be non-trivial.

Then there exists minimum \( i_k < j_m \) where \( i_k \theta j_m \) and \( k,m < 4 \). We claim \( \theta = Cg(0_k,1_k) \).

By Lemma 3.0.14, we have \( Cg(i_k,j_m) = Cg(0_k,1_k) \), so \( Cg(0_k,1_k) \subseteq \theta \). But any for \( (x,y) \in \theta \) we have \( x,y \geq i_k \), and so \( x,y \in [0_k,1_k]_L \), and thus by Lemma 1.2.9 we have \( (x,y) \in Cg(0_k,1_k) \). This gives \( \theta \subseteq Cg(0_k,1_k) \).

The other direction is clear.

We claim \( L_b \) is subdirectly irreducible if and only if \( b \notin B \).

(\( \Rightarrow \)) : Suppose \( b \notin B \).

Then for all \( i \), there are only finitely many \( j \) where \( P_b(i,j) = 1 \). Thus for any \( i \), \( c_s(i) \) is eventually constant in \( s \). Define \( c(i) = \lim_{s \to \infty} c_s(i) \). We then have that \( 0_{c(i)+1},1_{c(i)+1} \) satisfy the hypothesis of Theorem 1.2.8, so \( Cg(0_{c(i)+1},1_{c(i)+1}) \subseteq Cg(0_{c(i)},1_{c(i)}) \) for all \( i \in \omega \). So for any \( a \neq b \), there exists some maximal \( i \) where \( a,b \in [0_{c(i)},1_{c(i)}]_L \). Then \( (a,b) \notin Cg(0_{c(i)+1},1_{c(i)+1}) \), so thus no \( a,b \) can produce a monolith. Thus \( L_b \) is not subdirectly irreducible.

(\( \Leftarrow \)) : Suppose \( b \in B \). Then there exists a minimal \( i \) where there exist infinitely many \( j \) where \( P_b(i,j) = 1 \). Since \( i \) is minimal if \( i > 0 \), there exists a \( j_0 \) where for all \( j > j_0 \), \( P_b(i-1,j) = 0 \). Thus for all \( s > s_0 = \frac{j_0(j_0+1)}{2}+i-1 \), we have \( c_{s_0}(i-1) = c_s(i-1) \). We show \( Cg(0_{c_{s_0}(i-1)},1_{c_{s_0}(i-1)}) \) is a monolith of \( L_b \).

Suppose \( \theta \in \text{Con}(L_b) \) is nontrivial. Then there exists some \( a \neq b \) where \( (a,b) \in \theta \). By Lemma 3.0.19, \( Cg(a,b) = Cg(0_k,1_k) \) for some \( k \). If \( k < c_{s_0}(i-1) \), then
0_{c_{s_0}(i-1)}, 1_{c_{s_0}(i-1)} \in [0_k, 1_k)_L which gives by Lemma 1.2.9 that \((0_{c_{s_0}(i-1)}, 1_{c_{s_0}(i-1)}) \in Cg(0_k, 1_k) \subseteq \theta\). So suppose \(k > c_{s_0}(i-1)\). For infinitely many \(s\), \(c_{s+1}(i) = c_s(i + 1)\), so \(c_s(i)\) is unbounded. Thus there is an \(s_1 > s_0\) where \(c_{s_1}(i + 1)(i) > k\). But then by Lemma 3.0.18 \(Cg(0_{c_{s_0}(i-1)}, 1_{c_{s_0}(i-1)}) = Cg(0_k, 1_k)\), so \((0_{c_{s_0}(i-1)}, 1_{c_{s_0}(i-1)}) \in \theta\). Thus \(Cg(0_{c_{s_0}(i-1)}, 1_{c_{s_0}(i-1)})\) a the monolith of \(L_b\), so \(L_b\) is subdirectly irreducible.

Hence \(L_b\) is subdirectly irreducible if and only if \(b \in B\). \(\square\)
BIBLIOGRAPHY


