MATH 100 ♢ Survey of Mathematics ♢Spring 2012

Leonhard Euler and the Königsberg Bridges

The Königsberg Bridges over the river Pregel

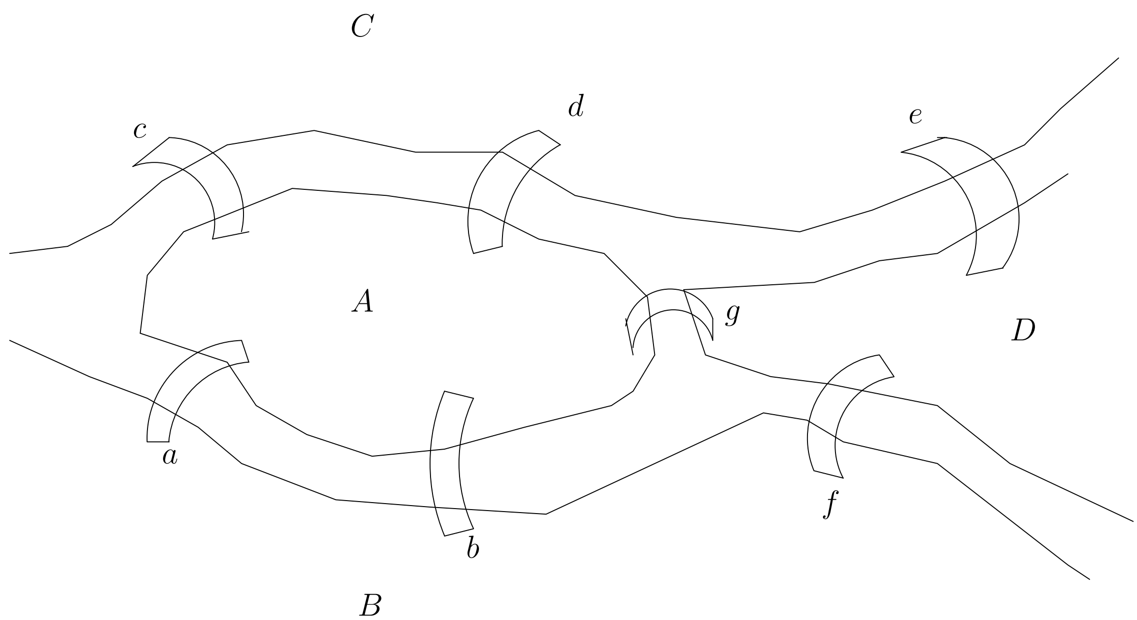

FIGURE 1. Bridges

Is there a path that crosses every bridge exactly once?

- **Leonhard Euler** (1707-1783)
- Born in Basel, Switzerland, died in St. Petersburg, Russia.
- Worked at Academies in St. Petersburg and Berlin.
- "Euler calculated without apparent effort, as men breathe, or as eagles sustain themselves in the wind." "Euler dashed off memoirs in half an hour between first and second calls to dinner."
- "Had 13 children and could do mathematics having children play around him and holding a baby on his lap."
- His eyesight weakened and eventually he was completely blind, but kept producing mathematics as before.
- Produced 800 memoirs and books.

Abstraction and mathematical modeling.

- What is relevant to the question?
- Which factors can be ignored in order to simplify the problem?
  1. Modeling the flight of the cannonball: air resistance, shape of the bullet ignored.
  2. Modeling robots
  3. Modeling economic developments: Variables are ignored, assumption are made. The mathematical predictions may come out wrong, not because of the mathematics but because of bad assumptions.

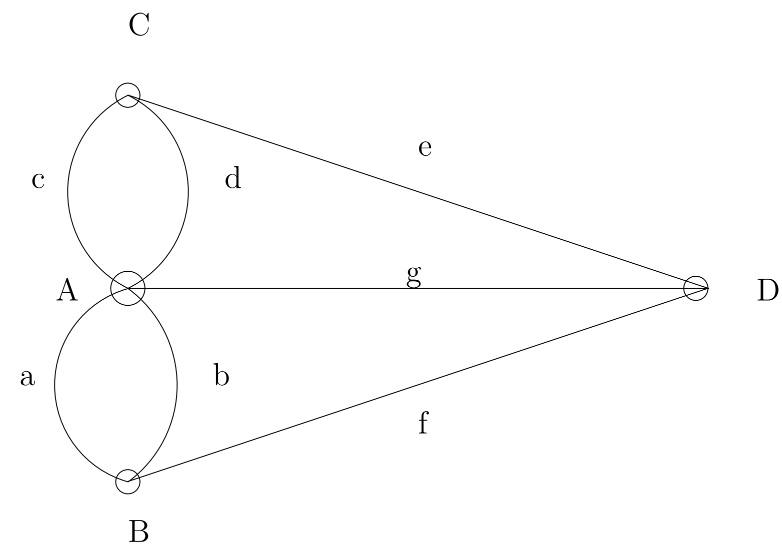

FIGURE 2. Euler's Model

Idea of checking all possibilities

- We call a path a "good" path if it crosses each bridge exactly once.
- Paths can be listed e.g. $abcdefg$, $abcdgef$, ... There are $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ possibilities. Does one of them work?

Euler's solution.

Can B, C, or D be points where a "good" path neither starts nor ends? How about A?

Conclusion. *A good path has one starting point and one terminal point. The three points B, C, or D on Euler's graph must be starting points or end points of a good path. This is impossible. There is no good path.*

A new field of mathematics is born: Graph Theory.

Definition. A ***graph*** consists of a (finite) set of points, called ***vertices*** and certain lines between vertices called ***edges***.

Do the following graphs have good paths?

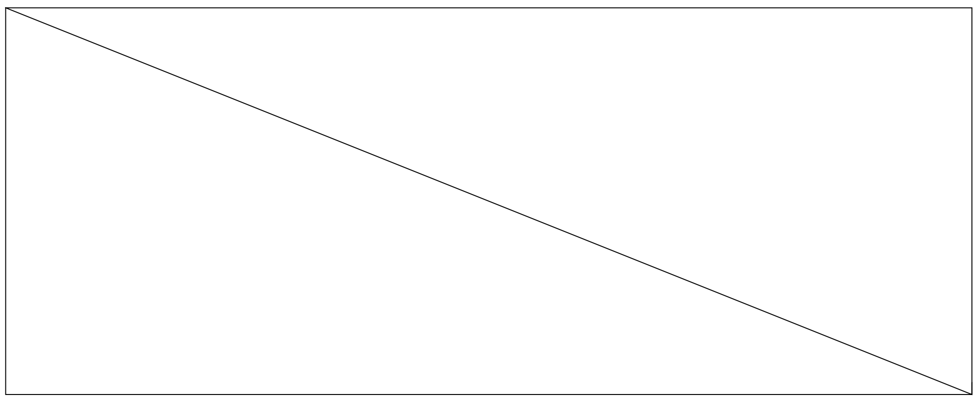

Figure 3

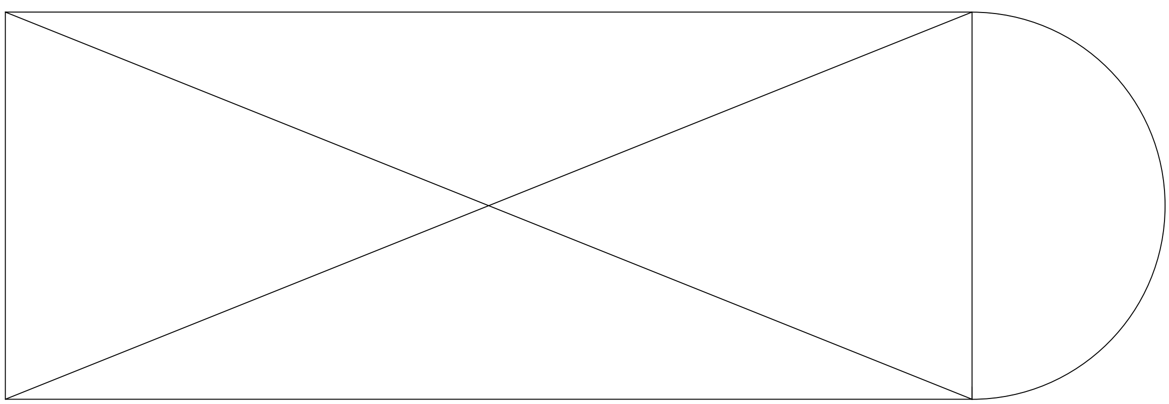

Figure 4

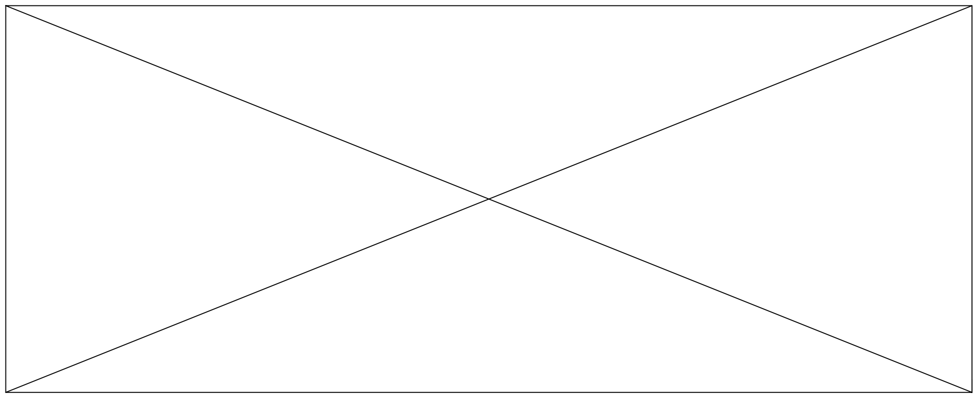

FIGURE 5

Theorems

Definition. The ***degree*** of a vertex in a graph is the number of edges ending (or starting) at the vertex.

Theorem. *If a vertex has odd degree, then a good path must either begin or end at this vertex.*

Theorem. *If the graph has more than two vertices of odd degree, then there is no good path.*

Theorem. *There is no graph with exactly one vertex of odd degree.*

Theorem. *If there are no vertices of odd degree, then there is a good path that can start at any vertex and must end at the same vertex.*

Applications

- The problem of the highway inspector.
- The problem of tracing a graph without lifting the pen.
- Transversing an apartment in such a way that we pass through every door exactly once.

What happens to the highway inspector if there are one-way streets?

What is graph theory?

- Describe and classify all graphs (too hard).
- Which important properties do graphs have? (degree, connectivity, completeness, loops)
- Describe and classify certain graphs.
- Graphs are used to present data in an efficient easy to read fashion: Organizational chart, networks, scheduling multitasking, precedence graphs, storing and retrieving data.

Other problems of graph theory

The problem of the traveling salesman. A salesman wishes to visit the towns of his district in such a way that he visits each town exactly once. Can he do it? How will he choose his route? This problem is not completely solved.

Dirac's Theorem. *Suppose that the district of the traveling salesman contains at least three towns and let n be the number of towns. If every town has at least $n/2$ streets ending in it, then there is a route for the traveling salesman.*

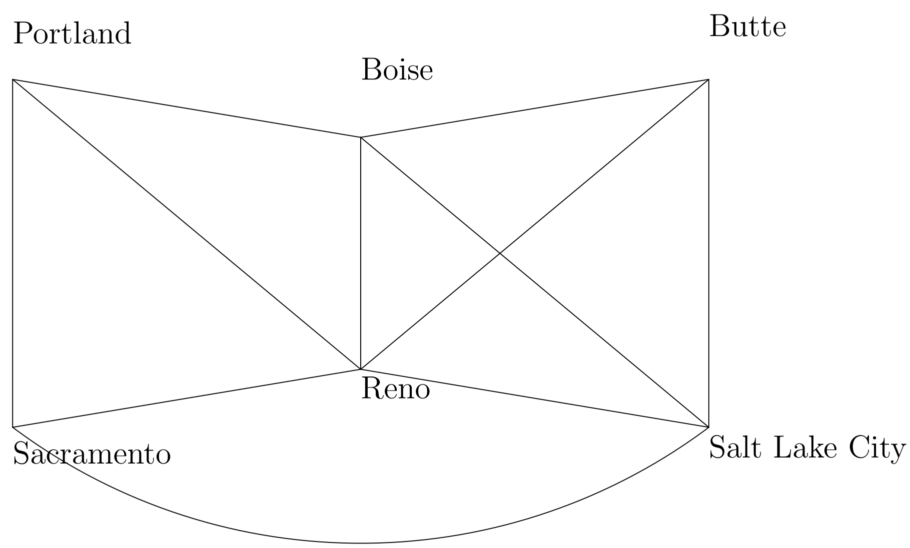

Figure 6

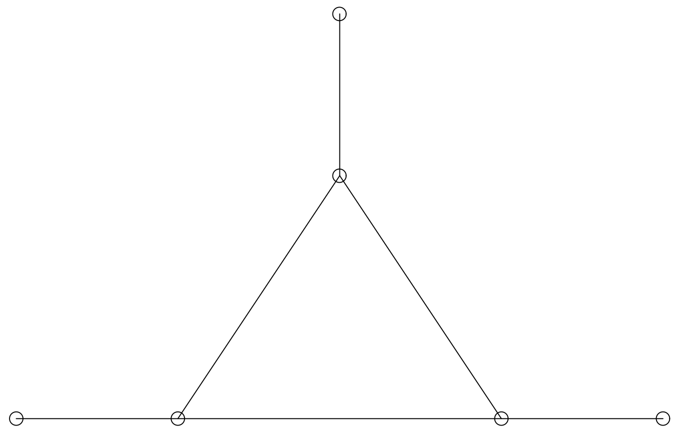

FIGURE 7

A shortest path problem. The diagram below shows a weighted simple graph. Look at the weight as distances. What is the shortest path from a to z?

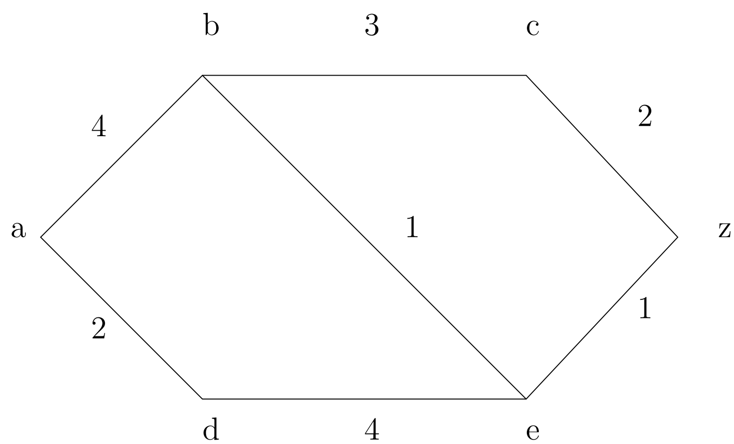

FIGURE 8

Edsgar Dijkstra's Algorithm for finding the shortest distance from a to z The graph has vertices $\nu_1, \nu_2, \ldots, \nu_n$ and weights $w(\nu_i, \nu_j)$ whose value is ∞ if there is no edge from ν_i to ν_j).

procedure
for $i := 1$ to n
$L(\nu_i) := \infty$
$L(a) := 0$
$S := \{\}$
while $z \notin S$
begin
$u :=$ a vertex not in S with $L(u)$ minimal
$S := S \cup \{u\}$
for all vertices not in S
if $L(u) + w(u, v) < L(v)$ **then** $L(v) := L(u) + w(u, v)$
end [$L(z)$ is the length of the shortest path.]

L(a)	L(b)	L(c)	L(d)	L(e)	L(z)	S	u
0	∞	∞	∞	∞	∞	$\{\}$	
	4	∞	2	∞	∞	$\{a\}$	a
	4	∞		6	∞	$\{a, d\}$	d
		7		5	∞	$\{a, b, d\}$	b
					6	$\{a, b, d, e\}$	e
					6	$\{a, b, d, e, z\}$	z

Shortest path is 6 units long.