Math 311 Midterm Review # 1

For the computational and example exercises, you do not need to justify your work. Simply write down the correct answer or produce a valid example and you will receive full credit.

For the proof exercises, you must produce a valid, clearly written proof. This must involve full sentences and be clear and easy to read. Partial credit will be assigned for proofs that do not fully meet these criteria.

1 Computational Exercises

Problem 1. There are three buslines: A, B and C. The combined ridership on lines A and B is 200 more than triple the ridership on line C. The difference between ridership on lines B and C is 2 times the ridership on line C. Also, the total ridership on all three lines is 1200 people per day. How many people ride each line each day?

Solution The system of equations is described by the augmented matrix

\[
\begin{bmatrix}
1 & 1 & -3 & 200 \\
0 & 1 & -3 & 0 \\
1 & 1 & 1 & 1200 \\
\end{bmatrix}
\]

Thus, the ridership on line A is 200, on line B is 750 and on line C is 250.

Problem 2. Consider the matrices \(A, B\) and \(C\).

\[
A = \begin{bmatrix}
1 & 1 & 2 & 0 \\
2 & 45 & 1 & -1 \\
3 & 2 & 0 & 0 \\
-1 & 1 & -2 & 0 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 0 \\
2 & 3 & \frac{1}{2} \\
-1 & -2 & -3 \\
\end{bmatrix}, \quad C = \begin{bmatrix}
2 & -1 & 0 \\
1 & 0 & -1 \\
-1 & -1 & 0 \\
\end{bmatrix}
\]

Calculate the following.

1. \(AB = \begin{bmatrix}
5 & 8 & 1 \\
5 & 9 & \frac{7}{2} \\
3 & 6 & 0 \\
-5 & -8 & -1 \\
\end{bmatrix}\)
2. $ABC = \begin{bmatrix} 17 & -6 & -8 \\ \frac{31}{2} & -\frac{17}{2} & -9 \\ 12 & -3 & -6 \\ -17 & 6 & 8 \end{bmatrix}$

3. $C^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{2}{3} \\ \frac{1}{3} & -1 & -\frac{1}{3} \end{bmatrix}$

4. $\det(A) = -12$

**Problem 3.** Consider the matrix

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & -2 & -1 \end{bmatrix}$$

Find linearly independent vectors that span $\text{null}(A)$.

**Solution** The reduced row echelon form of $D$ is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Thus,

$$\text{null}(D) = \left\{ \begin{bmatrix} x \\ -2x - y \\ 0 \\ x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\} = \text{span} \left( \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right)$$

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**2 Examples**

**Problem 4.** Give an example of a matrix, $A$, such that $A^T = A$. (This is called a symmetric matrix.)
Solution

\[ A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \]

\[ \square \]

**Problem 5.** Give an example of a matrix, \( A \), such that \( A^T = -A \). (This is called a skew-symmetric matrix.)

**Solution**

\[ A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

\[ \square \]

**Problem 6.** Give an example of a system of linear equations that is inconsistent.

**Solution**

\[ \begin{align*}
    x_1 + x_2 &= 0 \\
    x_1 + x_2 &= 1
\end{align*} \]

\[ \square \]

### 3 Proofs

**Problem 7.** Prove that if \( S = \{u_1, \ldots, u_n\} \) is linearly independent, then \( \text{span}(S) \neq \text{span}(u_2, \ldots, u_n) \).

**Solution** Clearly \( u_1 \in \text{span}(S) \). Suppose that \( u_1 \in \text{span}(u_2, \ldots, u_n) \), then there are scalars \( a_2, \ldots, a_n \) such that \( u_1 = a_2u_2 + \ldots + a_nu_n \). Thus, \( 0 = -u_1 + a_2u_2 + \ldots + a_nu_n \). This contradicts the assumption that \( S \) is linearly independent. Therefore, \( u_1 \notin \text{span}(u_2, \ldots, u_n) \).

\[ \square \]

**Problem 8.** Prove that if \( \text{null}(A) \neq \{0\} \) and \( \text{null}(B) = \{0\} \), then \( \text{null}(AB) \neq \{0\} \).

**Solution** Fix \( v \in \text{null}(A) \) such that \( v \neq 0 \). Since \( \text{null}(B) = \{0\} \), we know that the system of equation \( Bx = b \) has a unique solution for every vector \( b \). Let \( x_0 \) be such that \( Bx_0 = v \). We now have that

\[ (AB)x_0 = A(Bx_0) = Av = 0 \]

We conclude that \( x_0 \in \text{null}(AB) \) and that \( x_0 \neq 0 \).

\[ \square \]
**Problem 9.** Prove that for any matrix, $A$, the column space of $A$ is equal to the row space of $A^T$.

**Solution** Suppose $A$ is an $m \times n$ matrix with $ij$-entry $a_{ij}$. Then $A^T$ is an $n \times m$ matrix with $ij$-entry $a_{ji}$. Thus, the columns of $A$ are the vectors

$$\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$ 

Similarly, if we examine the matrix $A^T$, we see that

$$A^T = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}.$$ 

Thus, the rows of $A^T$ are

$$\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$ 

As these are exactly the same vectors as the columns of $A$, the two lists of vectors have the same span. In other words, the column space of $A$ is equal to the row space of $A^T$.

**Problem 10.** Prove that if $S$ is a set of vectors containing the zero vector, then $S$ is linearly dependent.

**Solution** Suppose $S = \{0, u_1, \ldots, u_n\}$, then $330 + 0u_1 + \cdots + 0u_n = 0$. Since the linear combination has a non-zero coefficient and sums to $0$, we have shown that $S$ is linearly dependent.