1 Homework

This section contains the assigned homework, due on 1/21/2019.

Problem 1 Prove that there is no uniform distribution on the sample space \( \Omega = \{0, 1, 2, 3, \ldots\} = \mathbb{N} \). (Recall: if \( m \) is a uniform distribution on a sample space \( \Omega \), then there is a fixed \( 0 \leq a \leq 1 \) such that \( m(\omega) = a \) for all \( \omega \in \Omega \).)

Solution Suppose \( m \) is a constant function such that \( m(n) = a \) for all \( n \in \mathbb{N} \). If \( a = 0 \), then \( \sum_{n=0}^{\infty} m(n) = 0 \), thus, \( m \) is not a distribution. If \( a > 0 \), since \( \sum_{n=0}^{\infty} m(n) = \sum_{n=0}^{\infty} a \) diverges, \( m \) cannot be a distribution.

2 Recommended Problems

These are the recommended problems. Two of them will appear on the quiz on 1/21/2019.

Problem 2 Suppose that \( \Omega \) is a finite sample space and there is a distribution defined on \( \Omega \). Prove that if \( A \) and \( B \) are events over the sample space \( \Omega \) with \( A \subseteq B \), then \( P(A) \leq P(B) \). Is it true that if \( A \subsetneq B \), then \( P(A) < P(B) \)? For the last question, either prove it is true or present a counterexample.

Solution Suppose \( A = \{\omega_1, \ldots, \omega_k\} \) and \( B = \{\omega_1, \ldots, \omega_k, \ldots, \omega_{k+n}\} \). Given a distribution function, \( m \), we have that

\[
P(A) = m(\omega_1) + \cdots + m(\omega_k) \leq m(\omega_1) + \cdots + m(\omega_k) + \cdots + m(\omega_{n+k}) = P(B)
\]

Thus, \( P(A) \leq P(B) \).

Consider the sample space \( \Omega = \{0, 1\} \) with distribution \( m \) such that \( m(0) = 1 \) and \( m(1) = 0 \). Let \( A = \{0\} \) and \( B = \{0, 1\} \). Although \( A \subsetneq B \), note that \( P(A) = m(0) = 1 = 1 + 0 = m(0) + m(1) = P(B) \). Thus, \( A \subsetneq B \) does not imply that \( P(A) < P(B) \).

Problem 3 Consider a well-shuffled standard deck of 52 cards. Let \( S \) be the event “the top card is a spade”, \( F \) is the event “the top card is a face card” and \( A \) is the event “the top card is an ace”.

1. Find \( P(S) \), \( P(F) \) and \( P(A) \).
2. Find $P(F \cup A)$, $P(S \cup F)$ and $P(S \cup A)$.

3. Find $P(F \cap A)$, $P(S \cap F)$ and $P(S \cap A)$.

4. Find $P(F \setminus S)$ and $P(S \setminus F)$.

Solution


2. $P(F \cup A) = (12 + 4)/52 = 4/13$, $P(S \cup F) = (10 + 3 + 9)/52 = 22/52 = 11/26$ and $P(S \cup A) = (12 + 1 + 3)/52 = 16/52 = 4/13$.

3. $P(F \cap A) = 0/52 = 0$, $P(S \cap F) = 3/52$ and $P(S \cap A) = 1/52$.

4. $P(F \setminus S) = (12 - 3)/52 = 9/52$ and $P(S \setminus F) = (13 - 3)/52 = 10/52$.

Problem 4 Suppose that $\Omega = \{a, b, c\}$ and $m$ is a distribution on $\Omega$ such that $m(a) = 1/4$ and $m(b) = 1/4$. Find the probabilities of all 8 different subsets of $\Omega$.

Solution Since $m(a) = 1/4$ and $m(b) = 1/4$, we know that $m(c) = 1/2$. Thus, $P(\emptyset) = 0$, $P(\{a\}) = 1/4$, $P(\{b\}) = 1/4$, $P(\{c\}) = 1/2$, $P(\{a, b\}) = 3/4$, $P(\{a, c\}) = 1/2$, $P(\{b, c\}) = 1/2$, and $P(\{a, b, c\}) = 1$.

Problem 5 Suppose a fair 6-sided die is rolled 3 times. What is the probability that the first roll is divisible by 2, the second roll is divisible by 3 and the third roll is divisible by 4?

Solution Let $A$ be the event that the first roll is divisible by 2, $B$ be the event that the second roll is divisible by 3 and let $C$ be the event that the third roll is divisible by 4. Since $\{2, 4, 6\}$ are divisible by 2, $P(A) = 1/2$. Since $\{3, 6\}$ are divisible by 3, $P(B) = 1/3$. Since $\{4\}$ is divisible by 4, $P(C) = 1/6$. So the probability that the first roll is divisible by 2, the second roll is divisible by 3 and the third roll is divisible by 4 is $(1/2)(1/3)(1/6) = 1/36$.

Problem 6 Suppose a fair coin is flipped 5 times. What is the probability that heads will come up exactly twice? (A tree diagram may help.)

Solution There are ten possibilities, thus, the probability is $10/32 = 5/16$. 

Problem 7 Suppose that 10% of the fish in a school are blue, 25% are in the family Acantharus and 30% are either blue or in the family Acantharus. What is the probability that a randomly selected fish is both blue and in the family Acantharus?

Solution Let $B$ be the event that the randomly selected fish is blue and let $A$ be the event that the randomly selected fish is in the family Acantharus. $P(B) = 0.1$, $P(A) = 0.25$ and $P(B \cup A) = 0.3$. Since

$$0.3 = P(B \cup A) = P(B) + P(A) - P(B \cap A) = 0.1 + 0.25 - P(B \cap A)$$

we have that $P(B \cap A) = 0.05$.

Problem 8 For each of problems 5, 6 and 7, state your answer using odds instead of probabilities.

Solution For problem 5, the odds are 1 to 35. For problem 6, the odds are 5 to 11. For problem 7, the odds are 1 to 19.

Problem 9 Suppose that $P(A) = P(B)$, $P(A \cup B) = p$ and $P(A \cap B) = q$. What is $P(A)$?

Solution Since $p = P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(A) - P(A \cap B) = 2P(A) - q$, we have that $P(A) = \frac{1}{2}(p + q)$.

Problem 10 Define a distribution, $m$, on the sample space $\Omega = \{a, b, c, d\}$ such that

1. $P(\{a, b, c\}) = \frac{3}{4}$
2. $P(\{a, b, d\}) = \frac{1}{2}$
3. $P(\{c\}) = \frac{1}{2}$.

Solution Define $m$ so that $m(a) = 0, m(b) = \frac{1}{4}, m(c) = \frac{1}{2}$ and $m(d) = \frac{1}{4}$.