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**Theorem on Local Extrema**

*If* $f(c)$ *is a local extremum, then either* $f$ *is not differentiable at* $c$ *or* $f'(c) = 0$.
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We will use this to prove

**Rolle’s Theorem**

*Let* \( a < b \). *If* \( f \) *is continuous on the closed interval* \([a, b]\) *and differentiable on the open interval* \((a, b)\) *and* \( f(a) = f(b) \), *then there is a* \( c \) *in* \((a, b)\) *with* \( f'(c) = 0 \). *That is, under these hypotheses, \( f \) has a horizontal tangent somewhere between* \( a \) *and* \( b \).
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*Let \( a < b \). If \( f \) is continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\) and \( f(a) = f(b) \), then there is a \( c \) in \((a, b)\) with \( f'(c) = 0 \). That is, under these hypotheses, \( f \) has a horizontal tangent somewhere between \( a \) and \( b \).*

Rolle’s Theorem, like the Theorem on Local Extrema, ends with \( f'(c) = 0 \). The proof of Rolle’s Theorem is a matter of examining cases and applying the Theorem on Local Extrema,
Proof of Rolle’s Theorem

We seek a $c$ in $(a, b)$ with $f'(c) = 0$. That is, we wish to show that $f$ has a horizontal tangent somewhere between $a$ and $b$. Keep in mind that $f(a) = f(b)$. 
Proof of Rolle’s Theorem

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Since $f$ is continuous on the closed interval $[a, b]$, the Extreme Value Theorem says that $f$ has a maximum value $f(M)$ and a minimum value $f(m)$ on the closed interval $[a, b]$. 
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First we suppose the maximum value $f(M) = f(m)$, the minimum value. So all values of $f$ on $[a, b]$ are equal, and $f$ is constant on $[a, b]$. 
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First we suppose the maximum value $f(M) = f(m)$, the minimum value. So all values of $f$ on $[a, b]$ are equal, and $f$ is constant on $[a, b]$. Then $f'(x) = 0$ for all $x$ in $(a, b)$. So one may take $c$ to be anything in $(a, b)$; for example, $c = \frac{a+b}{2}$ would suffice.
Proof of Rolle’s Theorem

Now we suppose \( f(M) \neq f(m) \). So at least one of \( f(M) \) and \( f(m) \) is not equal to the value \( f(a) = f(b) \).
Proof of Rolle’s Theorem

Now we suppose $f(M) \neq f(m)$. So at least one of $f(M)$ and $f(m)$ is not equal to the value $f(a) = f(b)$.

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Now we suppose \( f(M) \neq f(m) \). So at least one of \( f(M) \) and \( f(m) \) is not equal to the value \( f(a) = f(b) \).

We first consider the case where the maximum value \( f(M) \neq f(a) = f(b) \). So \( a \neq M \neq b \). But \( M \) is in \([a, b]\) and not at the end points.
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We first consider the case where the maximum value $f(M) \neq f(a) = f(b)$. So $a \neq M \neq b$. But $M$ is in $[a, b]$ and not at the end points. Thus $M$ is in the open interval $(a, b)$. 

So we are done with the proof of Rolle’s Theorem.
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The case with the minimum value \( f(m) \neq f(a) = f(b) \) is similar and left for you to do.

So we are done with the proof of Rolle’s Theorem.
We show that $x^5 + 4x = 1$ has exactly one solution.
joint application of Rolle’s Theorem and the Intermediate Value Theorem

We show that \( x^5 + 4x = 1 \) has exactly one solution. Let \( f(x) = x^5 + 4x \). Since \( f \) is a polynomial, \( f \) is continuous everywhere.

\( f'(x) = 5x^4 + 4 \geq 0 + 4 = 4 > 0 \) for all \( x \). So \( f'(x) \) is never 0. So by Rolle’s Theorem, no equation of the form \( f(x) = C \) can have 2 or more solutions.

In particular \( x^5 + 4x = 1 \) has at most one solution.

\( f(0) = 0^5 + 4 \cdot 0 = 0 < 1 < 5 = 1 + 4 = f(1) \).

Since \( f \) is continuous everywhere, by the Intermediate Value Theorem, \( f(x) = 1 \) has a solution in the interval \([0, 1]\).

Together these results say \( x^5 + 4x = 1 \) has exactly one solution, and it lies in \([0, 1]\).
We show that $x^5 + 4x = 1$ has exactly one solution. Let $f(x) = x^5 + 4x$. Since $f$ is a polynomial, $f$ is continuous everywhere. $f'(x) = 5x^4 + 4 \geq 0 + 4 = 4 > 0$ for all $x$. So $f'(x)$ is never 0.
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$f(0) = 0^5 + 4 \cdot 0 = 0 < 1 < 5 = 1 + 4 = f(1)$. Since $f$ is continuous everywhere, by the Intermediate Value Theorem, $f(x) = 1$ has a solution in the interval $[0, 1]$. Together these results say $x^5 + 4x = 1$ has exactly one solution, and it lies in $[0, 1]$. 

**joint application of Rolle’s Theorem and the Intermediate Value Theorem**
The traditional name of the next theorem is the Mean Value Theorem. A more descriptive name would be Average Slope Theorem.

**Mean Value Theorem**

*Let* \( a < b \). *If* \( f \) *is continuous on the closed interval* \([a, b]\) *and differentiable on the open interval* \((a, b)\), *then there is a* \( c \) *in* \((a, b)\) *with*

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f'(c) = \frac{f(b) - f(a)}{b - a}.
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That is, under appropriate smoothness conditions the slope of the curve at some point between $a$ and $b$ is the same as the slope of the line joining $\langle a, f(a) \rangle$ to $\langle b, f(b) \rangle$. The figure to the right shows two such points, each labeled $c$. 
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The Mean Value Theorem generalizes Rolle’s Theorem.
Let’s look again at the two theorems together.

**Rolle’s Theorem**
*Let* $a < b$. *If* $f$ *is continuous on* $[a, b]$ *and differentiable on* $(a, b)$ *and* $f(a) = f(b)$, *then there is a* $c$ *in* $(a, b)$ *with* $f'(c) = 0$.

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*Let* $a < b$. *If* $f$ *is continuous on* $[a, b]$ *and differentiable on* $(a, b)$, *then there is a* $c$ *in* $(a, b)$ *with*

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**Rolle’s Theorem**

Let $a < b$. If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$ and $f(a) = f(b)$, then there is a $c$ in $(a, b)$ with $f'(c) = 0$.

**Mean Value Theorem**

Let $a < b$. If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there is a $c$ in $(a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$ 

The proof of the Mean Value Theorem is accomplished by finding a way to apply Rolle’s Theorem. One considers the line joining the points $\langle a, f(a) \rangle$ and $\langle b, f(b) \rangle$. The difference between $f$ and that line is a function that turns out to satisfy the hypotheses of Rolle’s Theorem, which then yields the desired result.
Proof of the Mean Value Theorem

Suppose $f$ satisfies the hypotheses of the Mean Value Theorem.
Proof of the Mean Value Theorem

Suppose $f$ satisfies the hypotheses of the Mean Value Theorem. We let $g$ be the difference between $f$ and the line joining the points $\langle a, f(a) \rangle$ and $\langle b, f(b) \rangle$. That is, $g(x)$ is the height of the vertical green line in the figure to the right.
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The line joining the points $\langle a, f(a) \rangle$ and $\langle b, f(b) \rangle$ has equation

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$
Proof of the Mean Value Theorem

So

\[ g(x) = f(x) - \left[ f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right]. \]

\( g \) is the difference of two continuous functions. So \( g \) is continuous on \([a, b] \).
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\( g \) is the difference of two differentiable functions. So \( g \) is differentiable on \((a, b)\). Moreover, the derivative of \( g \) is the difference between the derivative of \( f \) and the derivative (slope) of the line.
Proof of the Mean Value Theorem

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\[ g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}. \]
Proof of the Mean Value Theorem

Both $f$ and the line go through the points $\langle a, f(a) \rangle$ and $\langle b, f(b) \rangle$. 
Proof of the Mean Value Theorem

Both $f$ and the line go through the points $\langle a, f(a) \rangle$ and $\langle b, f(b) \rangle$. So the difference between them is 0 at $a$ and at $b$. 
Proof of the Mean Value Theorem

Both \( f \) and the line go through the points \( \langle a, f(a) \rangle \) and \( \langle b, f(b) \rangle \). So the difference between them is 0 at \( a \) and at \( b \). Indeed,

\[
g(a) = f(a) - \left[ f(a) + \frac{f(b) - f(a)}{b - a} (a - a) \right] = f(a) - [f(a) + 0] = 0,
\]

and

\[
g(b) = f(b) - \left[ f(a) + \frac{f(b) - f(a)}{b - a} (b - a) \right]
\]

\[= f(b) - [f(a) + f(b) - f(a)] = 0.\]
So Rolle’s Theorem applies to $g$.
Proof of the Mean Value Theorem

So Rolle’s Theorem applies to $g$. So there is a $c$ in the open interval $(a, b)$ with $g'(c) = 0$. 

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g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.
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So Rolle's Theorem applies to \( g \). So there is a \( c \) in the open interval \((a, b)\) with \( g'(c) = 0\). Above we calculated that

\[
g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.
\]

Using that we have

\[
0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}
\]

which is what we needed to prove.
Example

We illustrate The Mean Value Theorem by considering $f(x) = x^3$ on the interval $[1, 3]$. 

$f(b) - f(a) = f(3) - f(1) = 27 - 1 = 13$.

$f'(c) = 3c^2$. So we seek a $c$ in $[1, 3]$ with $3c^2 = 13$. 
Example

We illustrate The Mean Value Theorem by considering $f(x) = x^3$ on the interval $[1, 3]$. 
$f$ is a polynomial and so continuous everywhere. For any $x$ we see that $f'(x) = 3x^2$. So $f$ is continuous on $[1, 3]$ and differentiable on $(1, 3)$. So the Mean Value theorem applies to $f$ and $[1, 3]$. 
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\[
\frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} = \frac{27 - 1}{2} = 13.
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$f'(c) = 3c^2$. So we seek a $c$ in $[1, 3]$ with $3c^2 = 13$. 
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\[ 3c^2 = 13 \iff c^2 = \frac{13}{3} \iff c = \pm \sqrt{\frac{13}{3}}. \]
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\[ 3c^2 = 13 \text{ iff } c^2 = \frac{13}{3} \text{ iff } c = \pm \sqrt{\frac{13}{3}}. \]

\[-\sqrt{\frac{13}{3}} \text{ is not in the interval } (1, 3), \text{ but } \sqrt{\frac{13}{3}} \text{ is a little bigger than } \sqrt{\frac{12}{3}} = \sqrt{4} = 2. \text{ So } \sqrt{\frac{13}{3}} \text{ is in the interval } (1, 3). \]
Example

\[ 3c^2 = 13 \iff c^2 = \frac{13}{3} \iff c = \pm\sqrt{\frac{13}{3}}. \]

\(-\sqrt{\frac{13}{3}}\) is not in the interval \((1, 3)\), but \(\sqrt{\frac{13}{3}}\) is a little bigger than \(\sqrt{\frac{12}{3}} = \sqrt{4} = 2\). So \(\sqrt{\frac{13}{3}}\) is in the interval \((1, 3)\).

So \(c = \sqrt{\frac{13}{3}}\) is in the interval \((1, 3)\), and

\[
f'(c) = f' \left( \sqrt{\frac{13}{3}} \right) = 13 = \frac{f(3) - f(1)}{3 - 1} = \frac{f(b) - f(a)}{b - a}.
\]