CONCAVITY AND GRAPHING

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We need some

**Definitions 1.** Let \( f \) be a function and \( I \) be an interval. \( f \) is *concave up* on \( I \) iff its derivative \( f' \) is increasing on \( I \). \( f \) is *concave down* on \( I \) iff its derivative \( f' \) is decreasing on \( I \). \( \langle a, f(a) \rangle \) is an *inflection point of* \( f \) iff there is a change in concavity from up to down or from down to up at \( a \).

So if a function is increasing and concave up on an interval, then the slopes of the tangents are positive and getting steeper. On the other hand, if a function is increasing and concave down on an interval, then the slopes of the tangents are positive but getting less steep. Etc.

The four different possibilities are pictured below.

![Graphs of Concavity](image)

Earlier we applied the Mean Value Theorem to prove the

**Theorem 1.** Suppose the function \( f \) is continuous on the interval \( I \) and differentiable on its interior \( I^0 \).

1. If \( f'(x) = 0 \) for all \( x \) in \( I^0 \), then \( f \) is constant on \( I \).
2. If \( f'(x) > 0 \) for all \( x \) in \( I^0 \), then \( f \) is increasing on \( I \).
3. If \( f'(x) < 0 \) for all \( x \) in \( I^0 \), then \( f \) is decreasing on \( I \).

If we apply this theorem to \( f' \) and \( f'' \) instead of \( f \) and \( f' \), we obtain results about concavity.

**Corollary 2.** Suppose \( f' \) is continuous on the interval \( I \) and differentiable on its interior \( I^0 \).

1. If \( f''(x) > 0 \) for all \( x \) in \( I^0 \), then \( f \) is concave up on \( I \).
2. If \( f''(x) < 0 \) for all \( x \) in \( I^0 \), then \( f \) is concave down on \( I \).

This gives us further information which we will apply below to the analysis of functions, but first we make one further general observation.
The definitions imply that at an inflection point for \( f \) we have \( f' \) switching from increasing to decreasing or \( f' \) is switching from decreasing to increasing. That is, at an inflection point for \( f \) we see that \( f' \) has either a local max or a local min. This means

**Proposition 3.** If \( \langle a, f(a) \rangle \) is an inflection point of \( f \), then either \( f''(a) = 0 \) or \( f''(a) \) does not exist.

**Warning.** It does not go the other way. It can happen that \( f''(a) = 0 \) even though \( \langle a, f(a) \rangle \) is not an inflection point of \( f \).

Let \( f(x) = x^4 \). Then \( f'(x) = 4x^3 \) which is a polynomial and continuous everywhere. Also, \( f''(x) = 12x^2 \). So \( f''(0) = 0 \), but \( f''(x) > 0 \) if \( x \neq 0 \). So \( f'(x) > 0 \) on \( (-\infty, 0) \) and on \( (0, +\infty) \). Then Corollary 2 implies \( f \) is concave up on \( (-\infty, 0] \) and on \( [0, +\infty) \). Thus \( f \) is concave up on \( (-\infty, +\infty) \). So \( f \) has no inflection points at all even though \( f''(0) = 0 \). The graph is just very flat near 0, but there is no concavity switch and no inflection point.

Now we will analyze and sketch some functions using the tools at hand. What we need to know in each case is where \( f \) and \( f' \) are continuous and where \( f' \) and \( f'' \) are positive and negative.

**Example 1.** Problem. \( f(x) = x^3 - 3x + 2 \). Find where \( f \) is increasing, decreasing, concave up and concave down. Find all local extrema and inflection points. Carefully justify each claim. Sketch \( f \) showing all these things.

\[ f'(x) = 3x^2 - 3, \quad \text{and} \quad f''(x) = 6x. \]

In earlier work we found that \( f \) is increasing on \( (-\infty, -1] \) and decreasing on \( [-1, 1] \) and increasing on \( [1, +\infty) \). This allowed us to conclude that \( f(-1) = 4 \) is a local maximum value and that \( f(1) = 0 \) is a local minimum value.

\[ f'(x) = 3x^2 - 3 \] is a polynomial and so continuous everywhere. If \( x < 0 \), then \( f''(x) = 6x < 0 \), and so by Corollary 2 we see that \( f \) is concave down on \( (-\infty, 0] \). If \( x > 0 \), then \( f''(x) = 6x > 0 \), and so by Corollary 2 we see that \( f \) is concave up on \( [0, +\infty) \).

Since concavity changes from down to up at 0, we have that \( \langle 0, f(0) \rangle \) is an inflection point. The graph follows.
The local extrema are indicated in blue and the inflection point in green.

**Example 2.** Problem. $f(x) = 3x^4 + 4x^3$. Find where $f$ is increasing, decreasing, concave up and concave down. Find all local extrema and inflection points. Carefully justify each claim. Sketch $f$ showing all these things.

$f(x)$ is a polynomial and so continuous everywhere.

$f'(x) = 12x^3 + 12x^2 = 12x^2(x + 1)$. Now $x + 1 < 0$ if $x < -1$, and $x + 1 > 0$ if $x > -1$. Also, $12x^2 > 0$, if $x \neq 0$. So $f'(x) < 0$ on $(-\infty, -1)$, and $f'(x) > 0$ on $(-1, 0)$ and on $(0, +\infty)$.

Now by Theorem 1 we see that $f$ is decreasing on $(-\infty, -1]$, and $f$ is increasing on $[-1, 0]$ and on $[0, +\infty)$. Because of the overlap, $f$ is increasing on $[-1, +\infty)$.

Because $f$ is decreasing on $(-\infty, -1]$ and increasing on $[-1, +\infty)$, we see that $f(-1) = -1$ is a local (and absolute) minimum value. There are no local maxima.

$f'(x)$ is a polynomial and so continuous everywhere.

$f''(x) = 36x^2 + 24x = 36x(x + \frac{2}{3})$.

Now if $x < -\frac{2}{3}$, then $x < x + \frac{2}{3} < 0$, and so $f''(x) = 36x(x + \frac{2}{3}) > 0$.

If $-\frac{2}{3} < x < 0$, then $x < 0 < x + \frac{2}{3}$, and so $f''(x) = 36x(x + \frac{2}{3}) < 0$.

If $0 < x$, then $0 < x < x + \frac{2}{3}$, and so $f''(x) = 36x(x + \frac{2}{3}) > 0$.

Summarizing, $f''(x) > 0$ on $(-\infty, -\frac{2}{3})$, and $f''(x) < 0$ on $(-\frac{2}{3}, 0)$, and $f''(x) > 0$ on $[0, +\infty)$. Corollary 2 implies that $f$ is concave up on $(-\infty, -\frac{2}{3}]$ and concave down on $[-\frac{2}{3}, 0]$ and concave up on $[0, +\infty)$. Because of the two concavity changes, $f$ has inflection points at $-\frac{2}{3}$ and at $0$. 


Example 3. Problem. $f(x) = x + x^{2/3}$. Find where $f$ is increasing, decreasing, concave up and concave down. Find all local extrema and inflection points. Carefully justify each claim. Sketch $f$ showing all these things.

$x^{2/3}$ is continuous on $(-\infty, +\infty)$ (see the appendix to these notes). So $f$ is a sum of two continuous functions and is thus continuous on $(-\infty, +\infty)$.

$f'(x) = 1 + \frac{2}{3}x^{-1/3}$, and $f''(x) = -\frac{2}{9}x^{-4/3}$.

Note that $f'(x)$ is not defined at 0, but $f'(x)$ is continuous everywhere else (again see the appendix).

$x^{-4/3} = (x^{-2/3})^2 = \left(\frac{1}{x^{2/3}}\right)^2$ which is positive for all $x \neq 0$. So $f''(x) < 0$ for all $x \neq 0$. Now by Corollary 2, $f$ is concave down on $(-\infty, 0)$ and on $(0, +\infty)$. (However, $f$ is not concave down on $(-\infty, 0) \cup (0, +\infty)$.) Since there are no changes in concavity, there are no inflection points.

$$f'(x) = 1 + \frac{2}{3}x^{-1/3} = \frac{x^{1/3} + \frac{2}{3}}{x^{1/3}}$$

$$= \frac{x^{1/3} + \frac{2}{3}}{x^{1/3}} \cdot \frac{(x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9}}{(x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9}} \cdot \frac{x^{2/3}}{x^{2/3}}$$

$$= \frac{x + \frac{8}{9}}{x} \cdot \frac{x^{2/3}}{(x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9}}.$$

$$(x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9} = (x^{1/3})^2 - 2\frac{1}{3}x^{1/3} + \frac{1}{3} + \frac{1}{3} = (x^{1/3} - \frac{1}{3})^2 + \frac{1}{3} \geq \frac{1}{3} > 0 \text{ for all } x. \text{ Also,}$$

$$x^{2/3} = (x^{1/3})^2 > 0 \text{ for all } x \neq 0.$$
So
\[ \frac{x^{2/3}}{(x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9}} > 0 \quad \text{for all } x \neq 0. \]

If \( x < -\frac{8}{27} \), then \( x < x + \frac{8}{27} < 0 \). So \( \frac{x + \frac{8}{27}}{x} > 0 \), and thus \( f'(x) > 0 \).

If \( -\frac{8}{27} < x < 0 \), then \( x < 0 < x + \frac{8}{27} \). So \( \frac{x + \frac{8}{27}}{x} < 0 \), and thus \( f'(x) < 0 \).

If \( 0 < x \), then \( 0 < x < x + \frac{8}{27} \). So \( \frac{x + \frac{8}{27}}{x} > 0 \), and thus \( f'(x) > 0 \).

Summarizing, \( f'(x) > 0 \) on \((-\infty, -\frac{8}{27})\), and \( f'(x) < 0 \) on \((-\frac{8}{27}, 0)\), and \( f'(x) > 0 \) on \((0, +\infty)\).

So by Theorem 1 we have that \( f \) is increasing on \((-\infty, -\frac{8}{27})\), and that \( f \) is decreasing on \([-\frac{8}{27}, 0]\), and that \( f \) is increasing on \([0, +\infty)\).

Because of this information, we have that \( f(-\frac{8}{27}) \) is a local max and \( f(0) \) is a local min.

\[ f(-\frac{8}{27}) = -\frac{8}{27} + (-\frac{8}{27})^{2/3} = -\frac{8}{27} + \frac{4}{9} = \frac{4}{27}, \text{ and } f(0) = 0. \]

The graph follows. The local extrema are indicated in blue.
Example 4. Problem. \( f(x) = (x^2 - 1)^2 \). Find where \( f \) is increasing, decreasing, concave up and concave down. Find all local extrema and inflection points. Carefully justify each claim. Sketch \( f \) showing all these things.

\( f(x) \) is a polynomial and so continuous everywhere.

\( f'(x) = 2(x^2 - 1)2x = 4x(x + 1)(x - 1) \).

If \( x < -1 \), then \( x - 1 < x < x + 1 < 0 \), and so \( f'(x) = 4(x - 1)x(x + 1) < 0 \); that is \( f'(x) \) is a product of three negative numbers and a positive number (4), and so it is negative.

If \( -1 < x < 0 \), then \( x - 1 < 0 < x + 1 \), and so \( f'(x) = 4(x - 1)x(x + 1) > 0 \); that is \( f'(x) \) is a product of one negative number and three positive numbers, and so it is negative.

If \( 0 < x < 1 \), then \( x - 1 < 0 < x < x + 1 \), and so \( f'(x) = 4(x - 1)x(x + 1) > 0 \); that is \( f'(x) \) is a product of three negative numbers and a positive number (4), and so it is negative.

Summarizing, \( f'(x) < 0 \) on \(( -\infty, -1) \), and \( f'(x) > 0 \) on \((-1, 0) \), and \( f'(x) < 0 \) on \((0, 1) \), and \( f'(x) > 0 \) on \((1, \infty) \).

Now by Theorem 1, the function \( f \) is decreasing on \((-\infty, -1) \), and \( f \) is increasing on \([-1, 0) \), and \( f \) is decreasing on \([0, 1] \), and \( f \) is increasing on \([1, \infty) \).

Since \( f \) is decreasing on \((-\infty, -1) \) and increasing on \([-1, 0) \), so \( f(-1) \) is a local minimum value.

Since \( f \) is increasing on \([-1, 0) \) and decreasing on \([0, 1] \), so \( f(0) \) is a local maximum value.

Since \( f \) is decreasing on \([0, 1] \) and increasing on \([1, \infty) \), so \( f(1) \) is a local minimum value.

\( f''(x) = (x^2 - 1)2x = 4(x^3 - x) \) is a polynomial and so continuous everywhere.

\( f''(x) = 4[3x^2 - 1] = 12(x^2 - \frac{1}{3}) = 12\left(x + \frac{1}{\sqrt{3}}\right)\left(x - \frac{1}{\sqrt{3}}\right) \).

If \( x < -\frac{1}{\sqrt{3}} \), then \( (x - \frac{1}{\sqrt{3}}) < (x + \frac{1}{\sqrt{3}}) < 0 \), and so \( f''(x) = 12\left(x - \frac{1}{\sqrt{3}}\right)\left(x + \frac{1}{\sqrt{3}}\right) > 0 \).

If \(-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \), then \( (x - \frac{1}{\sqrt{3}}) < 0 < (x + \frac{1}{\sqrt{3}}) \), and so \( f''(x) = 12\left(x - \frac{1}{\sqrt{3}}\right)\left(x + \frac{1}{\sqrt{3}}\right) < 0 \).

If \( \frac{1}{\sqrt{3}} < x \), then \( 0 < (x - \frac{1}{\sqrt{3}}) < (x + \frac{1}{\sqrt{3}}) \), and so \( f''(x) = 12\left(x - \frac{1}{\sqrt{3}}\right)\left(x + \frac{1}{\sqrt{3}}\right) > 0 \).

Summarizing, \( f''(x) > 0 \) on \((-\infty, -\frac{1}{\sqrt{3}}) \), and \( f''(x) < 0 \) on \(\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \), and \( f''(x) > 0 \) on \(\left(\frac{1}{\sqrt{3}}, \infty\right) \).

Now by Corollary 2, the function \( f \) is concave up on \((-\infty, -\frac{1}{\sqrt{3}}) \), and \( f \) is concave down on \(\left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \), and \( f \) is concave up on \(\left[\frac{1}{\sqrt{3}}, \infty\right) \).

Because of these changes in concavity, \( f \) has inflection points at \(\pm\frac{1}{\sqrt{3}} \).
The graph follows. The local extrema are indicated in blue and the inflection points in green.

Example 5. Problem. \( f(x) = x^4(x - 2)^4 \). Find where \( f \) is increasing, decreasing, concave up and concave down. Find all local extrema and inflection points. Carefully justify each claim. Sketch \( f \) showing all these things.

\( f \) is a polynomial and so continuous everywhere.

\[
f'(x) = 4x^3(x - 2)^4 + x^4 4(x - 2)^3 = 4x^3(x - 2)^3[x - 2 + x] = 4x^3(x - 2)^3[2x - 2] = 8x^3(x - 2)^3(x - 1).
\]

If \( x < 0 \), then \( x - 2 < x - 1 < x < 0 \), and so \( f'(x) = 8(x - 2)^3(x - 1)x^3 < 0 \).
If \( 0 < x < 1 \), then \( x - 2 < x - 1 < 0 < x \), and so \( f'(x) = 8(x - 2)^3(x - 1)x^3 > 0 \).
If \( 1 < x < 2 \), then \( x - 2 < 0 < x - 1 < x \), and so \( f'(x) = 8(x - 2)^3(x - 1)x^3 < 0 \).
If \( 2 < x \), then \( 0 < x - 2 < x - 1 < x \), and so \( f'(x) = 8(x - 2)^3(x - 1)x^3 > 0 \).

Summarizing, \( f'(x) < 0 \) on \((-\infty, 0)\), and \( f'(x) > 0 \) on \((0, 1)\), and \( f'(x) < 0 \) on \((1, 2)\), and \( f'(x) > 0 \) on \((2, +\infty)\).

Now by Theorem 1, we see that \( f \) is decreasing on \((-\infty, 0)\), and \( f \) is increasing on \([0, 1]\), and \( f \) is decreasing on \([1, 2]\), and \( f \) is increasing on \([2, +\infty)\).

So \( 0 = f(0) \) is a local minimum value, and \( 1 = f(1) \) is a local maximum value, and \( 0 = f(2) \) is a local minimum value.
Now we calculate and factor the second derivative.

\[
f''(x) = 24x^2(x - 2)^3(x - 1) + 24x^3(x - 2)^2(x - 1) + 8x^3(x - 2)^3
\]

\[
= 8x^2(x - 2)^2[3(x - 2)(x - 1) + 3(x - 1) + x(x - 2)]
\]

\[
= 8x^2(x - 2)^2[3(x^2 - 3x + 2) + 3(x^2 - x) + (x^2 - 2x)]
\]

\[
= 8x^2(x - 2)^2[7x^2 - 14x + 6] = 8x^2(x - 2)^2[7x^2 - 14x + 7 - 1]
\]

\[
= 56x^2(x - 2)^2 \left( x^2 - 2x + 1 - \frac{1}{7} \right) = 56x^2(x - 2)^2 \left( x - 1 \right)^2 - \frac{1}{7}
\]

\[
= 56x^2(x - 2)^2 \left( x - 1 - \frac{1}{\sqrt{7}} \right) \left( x - 1 + \frac{1}{\sqrt{7}} \right).
\]

\(f'\) is a polynomial and so continuous everywhere.

56\(x^2(x - 2)^2 > 0\) as long as \(0 \neq x \neq 2\).

If \(x < 1 - \frac{1}{\sqrt{7}}\), then \(x - 1 - \frac{1}{\sqrt{7}} < x - 1 + \frac{1}{\sqrt{7}} < 0\). So

\[
f''(x) = 56x^2(x - 2)^2 \left( x - 1 - \frac{1}{\sqrt{7}} \right) \left( x - 1 + \frac{1}{\sqrt{7}} \right) > 0\] as long as \(x \neq 0\).

If \(1 - \frac{1}{\sqrt{7}} < x < 1 + \frac{1}{\sqrt{7}}\), then \(0 \neq x \neq 2\) and \(x - 1 - \frac{1}{\sqrt{7}} < 0 < x - 1 + \frac{1}{\sqrt{7}}\). So

\[
f''(x) = 56x^2(x - 2)^2 \left( x - 1 - \frac{1}{\sqrt{7}} \right) \left( x - 1 + \frac{1}{\sqrt{7}} \right) < 0.
\]

If \(1 + \frac{1}{\sqrt{7}} < x\) and \(x \neq 2\), then \(0 < x - 1 - \frac{1}{\sqrt{7}} < x - 1 + \frac{1}{\sqrt{7}}\). So

\[
f''(x) = 56x^2(x - 2)^2 \left( x - 1 - \frac{1}{\sqrt{7}} \right) \left( x - 1 + \frac{1}{\sqrt{7}} \right) > 0\] as long as \(x \neq 2\).

Summarizing, \(f''(x) > 0\) on \((-\infty, 0)\) and on \((0, 1 - \frac{1}{\sqrt{7}})\), and \(f''(x) < 0\) on \((1 - \frac{1}{\sqrt{7}}, 1 + \frac{1}{\sqrt{7}})\), and \(f''(x) > 0\) on \(\left(1 + \frac{1}{\sqrt{7}}, 2\right)\) and on \((2, +\infty)\).

So by Corollary 2, we see that \(f\) is concave up on \((-\infty, 0)\) and on \([0, 1 - \frac{1}{\sqrt{7}}]\), and \(f\) is concave down on \(\left[1 - \frac{1}{\sqrt{7}}, 1 + \frac{1}{\sqrt{7}}\right]\), and \(f\) is concave up on \(\left[1 + \frac{1}{\sqrt{7}}, 2\right]\) and on \([2, +\infty)\).

Because of the overlaps, \(f\) is concave up on \((-\infty, 1 - \frac{1}{\sqrt{7}})\) and on \(\left[1 + \frac{1}{\sqrt{7}}, \infty\right)\).

Summarizing, \(f\) is concave up on \((-\infty, 1 - \frac{1}{\sqrt{7}})\) and concave down on \(\left[1 - \frac{1}{\sqrt{7}}, 1 + \frac{1}{\sqrt{7}}\right]\) and concave up on \(\left[1 + \frac{1}{\sqrt{7}}, \infty\right)\). This means that \(f\) has inflection points at \(1 \pm \frac{1}{\sqrt{7}}\).

\(f\) does not have inflection points at 0 or 2 even though \(f''(0) = 0 = f''(2)\). This is because the concavity does not change up to down or down to up at either 0 or 2.

Remember this example, and do not ever assert that \(a\) is an inflection point because \(f''(a) = 0\)!
The graph follows. The local extrema are indicated in blue and the inflection points in green.

Example 6. Problem. \( f(x) = \frac{2x^2 + 1}{x^2 - 1} \). Find where \( f \) is increasing, decreasing, concave up and concave down. Find all local extrema and inflection points. Find any horizontal and vertical asymptotes. Carefully justify each claim. Sketch \( f \) showing all these things.

\[
\begin{align*}
f(x) &= \frac{2x^2 + 1}{x^2 - 1} = \frac{2x^2 - 2 + 3}{x^2 - 1} = 2 + \frac{3}{x^2 - 1} \\
f'(x) &= \frac{-3}{(x^2 - 1)^2} \cdot 2x = 6 \frac{-x}{(x^2 - 1)^2} \\
f''(x) &= 6 \frac{-1(x^2 - 1)^2 - (-x)2(x^2 - 1)2x}{(x^2 - 1)^4} = 6 \frac{-x^2 + 1 + 4x^2}{(x^2 - 1)^3} = 6 \frac{3x^2 + 1}{(x^2 - 1)^3}
\end{align*}
\]

\( f(x) \) is not defined at \( \pm 1 \). \( f \) is a rational function. So \( f \) is continuous wherever it is defined. So \( f \) is continuous on \((-\infty, -1)\) and \((-1, 1)\) and on \((1, +\infty)\). \( f' \) is continuous on these same intervals for the same reasons.

\( (x^2 - 1)^2 > 0 \) as long as \( x \neq \pm 1 \). So if \( -1 \neq x < 0 \), then \( -x > 0 \) and \( f'(x) > 0 \). Theorem 1 implies \( f \) is increasing on \((-\infty, -1)\) and on \((-1, 0]\) (but not on \((-\infty, -1) \cup (-1, 0]\).)

\( (x^2 - 1)^2 > 0 \) as long as \( x \neq \pm 1 \). So if \( 1 \neq x > 0 \), then \( -x < 0 \) and \( f'(x) < 0 \). Theorem 1 implies \( f \) is decreasing on \([0, 1)\) and on \((1, +\infty)\) (but not on \([0, 1) \cup (1, +\infty)\).)

Because \( f \) is increasing on \((-1, 0]\) and decreasing on \([0, 1)\), we see that \( f(0) = -1 \) is a local maximum value. There are no local minima.
3x^2 + 1 \geq 1 > 0 \text{ for all } x. \text{ So the sign of } f''(x) \text{ depends only on the sign of } (x^2 - 1)^3 \text{ which is the same as the sign of } x^2 - 1. \\
If x < -1, then x - 1 < x + 1 < 0, and so x^2 - 1 = (x - 1)(x + 1) > 0, and so f''(x) > 0. \\
If -1 < x < 1, then x - 1 < 0 < x + 1, and so x^2 - 1 = (x - 1)(x + 1) < 0, and so f''(x) < 0. \\
If 1 < x, then 0 < x - 1 < x + 1, and so x^2 - 1 = (x - 1)(x + 1) > 0, and so f''(x) > 0. \\
Corollary 2 implies f is concave up on (-\infty, -1) and concave down on (-1, 1) and concave up on (1, +\infty). There are no inflection points.

\[
\lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \frac{2x^2 + 1}{x^2 - 1} = \lim_{x \to \pm \infty} \frac{2 + \frac{1}{x^2}}{1 - \frac{1}{x^2}} = \frac{2 + 0}{1 - 0} = 2.
\]
So \(y = 2\) is a horizontal asymptote.

\[
\lim_{x \to \pm 1} x^2 - 1 = 0 \neq 3 = \lim_{x \to \pm 1} 2x^2 + 1, \quad \text{and so } \lim_{x \to \pm 1} \left| \frac{2x^2 + 1}{x^2 - 1} \right| = \infty.
\]
So \(x = -1\) and \(x = 1\) are vertical asymptotes.

We can further analyze the vertical asymptotes. Note that \(2x^2 + 1 \geq 1 > 0\) for all \(x\). Above we showed that \(x^2 - 1 > 0\) on \((-\infty, -1)\) and on \((1, +\infty)\) while \(x^2 - 1 < 0\) on \((-1, 1)\). So

\[
\lim_{x \to -1^-} \frac{2x^2 + 1}{x^2 - 1} = \infty = \lim_{x \to -1^+} \frac{2x^2 + 1}{x^2 - 1}, \quad \text{and } \lim_{x \to -1^+} \frac{2x^2 + 1}{x^2 - 1} = -\infty = \lim_{x \to -1^-} \frac{2x^2 + 1}{x^2 - 1}.
\]

The graph follows in red with the asymptotes in green.
Here we discuss the continuity of $\sqrt[3]{x} = x^{1/3}$ and related functions.

Let $\varepsilon > 0$ and $\delta = \varepsilon^3$, and suppose $0 < |x - 0| < \delta$. We want $|x^{1/3} - 0| < \varepsilon$. Suppose instead that doesn’t happen. Then $|x|^{1/3} = |x^{1/3} - 0| \geq \varepsilon$. Using properties of inequalities we have $|x| = (|x|^{1/3})^3 \geq \varepsilon^3$. But we know $|x| = |x - 0| < \delta = \varepsilon^3$ not $\geq \varepsilon^3$. So it can’t be correct that $|x|^{1/3} \geq \varepsilon$. So it must be the case that $|x^{1/3} - 0| = |x|^{1/3} < \varepsilon$, as needed.

So by definition, $\lim_{x \to 0} x^{1/3} = 0 = 0^{1/3}$. So $\sqrt[3]{x} = x^{1/3}$ is continuous at 0.

$$\frac{d x^{1/3}}{dx} = \frac{1}{3} x^{-2/3} = \frac{1}{3(x^{1/3})^2}.$$ 

So $\sqrt[3]{x} = x^{1/3}$ is differentiable at every $x \neq 0$, and so $\sqrt[3]{x} = x^{1/3}$ is continuous at every $x \neq 0$. Putting all this together we see that $\sqrt[3]{x} = x^{1/3}$ is continuous on $(-\infty, +\infty)$.

Thus $\frac{1}{\sqrt[3]{x}} = \frac{1}{x^{1/3}} = x^{-1/3}$ is continuous wherever it is defined, which is all $x \neq 0$. So $\frac{2}{3}x^{-1/3}$ is also continuous for all $x \neq 0$. Since the horizontal straight line at 1 is continuous everywhere, $1 + \frac{2}{3}x^{-1/3}$ is continuous at all $x \neq 0$.

$x^{2/3} = (x^{1/3})^2$ is the composition of the two continuous functions $x^2$ and $x^{1/3}$. So $x^{2/3}$ is continuous everywhere. Now $x + x^{2/3}$ is continuous everywhere because it is the sum of two continuous functions.