APPLICATIONS OF THE MEAN VALUE THEOREM

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Definition 1. Let f be a function and S be a set of numbers. We say f is *increasing* on S iff $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ and x_1, x_2 are in S. We say f is *decreasing* on S iff $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ and x_1, x_2 are in S.

Definition 2. We define the *interior* I^o of an interval I as follows, where $a < b$.

$$\begin{aligned} [a,b]^o &= (a,b) & \qquad (a,b)^o &= (a,b) \\ (a,b]^o &= (a,b) & \qquad [a,b)^o &= (a,b) \\ (-\infty,b]^o &= (-\infty,b) & \qquad [a,\infty)^o &= (a,\infty) \\ (-\infty,b)^o &= (-\infty,b) & \qquad (a,\infty)^o &= (a,\infty) \end{aligned}$$

Theorem 1. *Suppose the function f is continuous on the interval I and differentiable on its interior I^o.*

(1) *If $f'(x) = 0$ for all x in I^o, then f is constant on I.*
(2) *If $f'(x) > 0$ for all x in I^o, then f is increasing on I.*
(3) *If $f'(x) < 0$ for all x in I^o, then f is decreasing on I.*

Proof. We suppose the function f is continuous on the interval I and differentiable on its interior I^o.

Case 1. Here we suppose $f'(x) = 0$ for all x in I^o. Let x_1, x_2 be in I with $x_1 < x_2$. So the closed interval $[x_1, x_2]$ is contained in I, and the open interval (x_1, x_2) is contained in I^o. So f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). So the Mean Value Theorem applies. So there is c between x_1 and x_2, with

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) = 0$$

and we get the last equality since c is in I^o since c is between x_1 and x_2, and they are in the interval I^o. So $f(x_2) - f(x_1) = 0$ and $f(x_2) = f(x_1)$. Since x_1 and x_2 were arbitrary members of I with $x_1 < x_2$, this means f is constant on I.

Case 2. Here we suppose $f'(x) > 0$ for all x in I^o. Let x_1, x_2 be in I with $x_1 < x_2$. (We aim to show $f(x_1) < f(x_2)$.) So the closed interval $[x_1, x_2]$ is contained in I, and the open interval (x_1, x_2) is contained in I^o. So f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). So the Mean Value Theorem applies. So there is c between x_1 and x_2, with

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0$$

and we get the last inequality since c is in I^o as above. Since $x_2 - x_1 > 0$, we multiply the inequality by $x_2 - x_1$, and we now have $f(x_2) - f(x_1) > 0$ and $f(x_2) > f(x_1)$. Since x_1 and x_2 were arbitrary members of I with $x_1 < x_2$, this means f is increasing on I.

Case 3. Here we suppose $f'(x) < 0$ for all x in I^o. Let x_1, x_2 be in I with $x_1 < x_2$. (We aim to show $f(x_1) > f(x_2)$.) So the closed interval $[x_1, x_2]$ is contained in I, and the open interval (x_1, x_2) is contained in I^o. So f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). So the Mean Value Theorem applies. So there is c between x_1 and x_2, with

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) < 0$$

and we get the last inequality since c is in I^o as above. Since $x_2 - x_1 > 0$, we multiply the inequality by $x_2 - x_1$, and we now have $f(x_2) - f(x_1) < 0$ and $f(x_2) < f(x_1)$. Since x_1 and x_2 were arbitrary members of I with $x_1 < x_2$, this means f is decreasing on I.

This ends the proof of the theorem. □

Remark. If there are numbers a, b, c, with $a < c < b$ and such that f is increasing on $(a, c]$ and decreasing on $[c, b)$, then $f(c)$ is a *local maximum value*!

If there are numbers a, b, c, with $a < c < b$ and such that f is decreasing on $(a, c]$ and increasing on $[c, b)$, then $f(c)$ is a *local minimum value*!

Example 1. Problem. $f(x) = x^3 - 3x + 2$. Find where f is increasing and decreasing. Find all local extrema. Sketch the graph.

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x + 1)(x - 1).$$

So $f'(x) = 0$ iff $x = -1$ or 1.
If $x < -1$, then $x - 1 < x + 1 < 0$, and so $f'(x) = 3(x + 1)(x - 1) > 0$.
If $-1 < x < 1$, then $x - 1 < 0 < x + 1$, and so $f'(x) = 3(x + 1)(x - 1) < 0$.
If $1 < x$, then $0 < x - 1 < x + 1$, and so $f'(x) = 3(x + 1)(x - 1) > 0$.
Now the Theorem tells us that
f is increasing on $(-\infty, -1]$,
f is decreasing on $[-1, 1]$, and
f is increasing on $[1, +\infty)$.
This latter information tells us that
$f(-1)$ is a local maximum value, and
$f(1)$ is a local minimum value.
$f(-1) = -1 + 3 + 2 = 4$ and $f(1) = 1 - 3 + 2 = 0$.

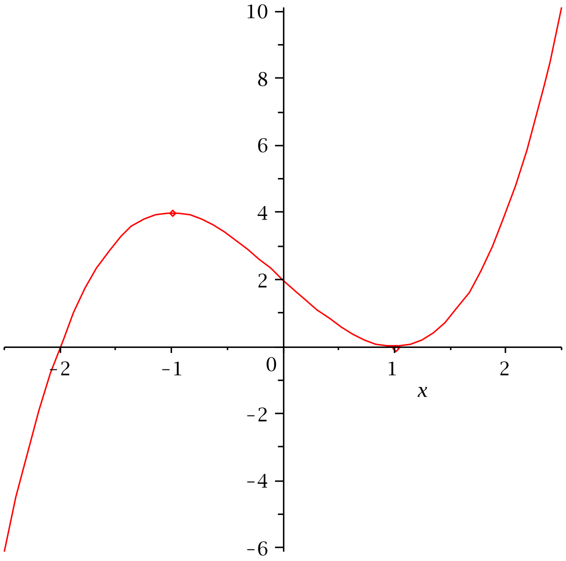

FIGURE 1. The graph of $x^3 - 3x + 2$

Example 2. If $f(x) = \frac{1}{x}$, then $f'(x) = \frac{-1}{x^2} < 0$ for all $x \neq 0$. f is continuous on $(-\infty, 0)$ and on $(0, \infty)$. So by the Theorem, f is decreasing on $(-\infty, 0)$ and on $(0, \infty)$.

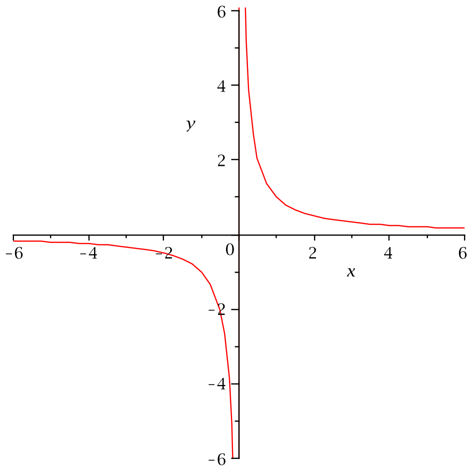

FIGURE 2. The graph of $1/x$

Warning. $f(x) = \frac{1}{x}$ is **NOT** decreasing on $(-\infty, 0) \cup (0, \infty)$ because, for example, $f(-1) = -1 < 1 = f(1)$, but if it were decreasing on $(-\infty, 0) \cup (0, \infty)$ we should instead have $f(-1) > f(1)$.

You know that

$$2x = \frac{d}{dx}(x^2) = \frac{d}{dx}(x^2 - 1) = \frac{d}{dx}(x^2 + 17) = \frac{d}{dx}(x^2 + \sqrt{2}) = \frac{d}{dx}(x^2 + C)$$

where C is any constant. Is there any other sort of solution to

$$\frac{dy}{dx} = 2x?$$

The next Theorem says, “No, there is not.”

Theorem 2. *Suppose the functions f and g are continuous on the interval I. If $f'(x) = g'(x)$ for all x in its interior I^o, then there is a constant C so that*

$$g(x) = f(x) + C \qquad \text{for all } x \text{ in } I.$$

Proof. Suppose f and g are as in the hypotheses. Set $h(x) = g(x) - f(x)$. Then h is continuous on I, and $h'(x) = g'(x) - f'(x) = f'(x) - f'(x) = 0$ for any x in I^o. Now by (1) of Theorem 1 h is constant on I; i.e., there is a constant C so that $g(x) - f(x) = h(x) = C$ for all x in I. That is, $g(x) = f(x) + C$ for all x in I. ☐

Definition 3. F is an *antiderivative* of f on the interval I if F is continuous on I and

$$F'(x) = f(x) \qquad \text{for all } x \text{ in } I^o.$$

Proposition 4. *Any antiderivative of x^n equals*

$$\frac{x^{n+1}}{n+1} + C, \quad \text{for } n \neq -1$$

for some constant C.

Proof.

$$\frac{d\left(\frac{x^{n+1}}{n+1}\right)}{dx} = \frac{1}{n+1}(n+1)x^{n+1-1} = x^n.$$

Now apply Theorem 2. ☐

Example 3. Any antiderivative of x^4 equals $\frac{x^5}{5} + C$ for some constant C, and any antiderivative of x^{-4} equals $\frac{x^{-3}}{-3} + C$ for some constant C.

Example 4. Any antiderivative of $\cos x$ equals $\sin x + C$ for some constant C, and any antiderivative of $\sin x$ equals $-\cos x + C$ for some constant C because
$\frac{d}{dx}(-\cos x) = - - \sin x = \sin x$.

Proposition 5. *If F and G are antiderivatives of f and g, respectively, and c is a real number, then*

- *$F + G$ is an antiderivative of $f + g$, and*
- *cF is an antiderivative of cf.*

Proof.

$$\frac{d}{dx}(F(x)+G(x)) = F'(x)+G'(x) = f(x)+g(x) \quad \text{and} \quad \frac{d}{dx}(cF(x)) = cF'(x) = cf(x).$$

□

Example 5. Problem: Given $f'(x) = 5x^4 + 6x^3 - 5x^2 + 11$ and $f(0) = -3$, find f.

Using the previous two Propositions (and Theorem 2), we see that

$$f(x) = x^5 + \frac{6x^4}{4} - \frac{5x^3}{3} + 11x + C \quad \text{and} \quad -3 = f(0) = 0 + 0 - 0 + 0 + C = C$$

and so $\quad f(x) = x^5 + \frac{3}{2}x^4 - \frac{5}{3}x^3 + 11x - 3.$

Now we use the above propositions to derive some laws of physics about a freely falling body near the earth's surface.

In what follows a is acceleration, v is velocity, and p is position.

We start with the understanding that a freely falling body near the earth's surface experiences a constant acceleration of 32 feet per second per second. That is,

$$a(t) = -32, \quad \text{but} \quad v'(t) = a(t).$$

So

$$v(t) = -32t + C.$$

Since $v(0) = 0 + C = C$, we replace C by the constant v_0 (which denotes initial velocity.) That is,

$$v(t) = -32t + v_0, \quad \text{but} \quad p'(t) = v(t).$$

So

$$p(t) = -16t^2 + v_0 t + D.$$

Since $p(0) = 0 + D = D$, we replace D by the constant p_0 (which denotes initial position.) That is,

$$p(t) = -16t^2 + v_0 t + p_0$$

where v_0 is the velocity at time 0 and p_0 is the position at time 0.