APPLICATIONS OF THE MEAN VALUE THEOREM

WILLIAM A. LAMPE

Definition 1. Let $f$ be a function and $S$ be a set of numbers. We say $f$ is increasing on $S$ iff $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ and $x_1, x_2$ are in $S$. We say $f$ is decreasing on $S$ iff $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ and $x_1, x_2$ are in $S$.

Definition 2. We define the interior $I^o$ of an interval $I$ as follows, where $a < b$.

\[
\begin{align*}
[a, b]^o &= (a, b) \\
(a, b]^o &= (a, b) \\
(-\infty, b]^o &= (-\infty, b) \\
(-\infty, b) &= (-\infty, b) \\
(a, \infty)^o &= (a, \infty) \\
[a, \infty)^o &= (a, \infty)
\end{align*}
\]

Theorem 1. Suppose the function $f$ is continuous on the interval $I$ and differentiable on its interior $I^o$.

1. If $f'(x) = 0$ for all $x$ in $I^o$, then $f$ is constant on $I$.
2. If $f'(x) > 0$ for all $x$ in $I^o$, then $f$ is increasing on $I$.
3. If $f'(x) < 0$ for all $x$ in $I^o$, then $f$ is decreasing on $I$.

Proof. We suppose the function $f$ is continuous on the interval $I$ and differentiable on its interior $I^o$.

Case 1. Here we suppose $f'(x) = 0$ for all $x$ in $I^o$. Let $x_1, x_2$ be in $I$ with $x_1 < x_2$. So the closed interval $[x_1, x_2]$ is contained in $I$, and the open interval $(x_1, x_2)$ is contained in $I^o$. So $f$ is continuous on $[x_1, x_2]$ and differentiable on $(x_1, x_2)$. So the Mean Value Theorem applies. So there is $c$ between $x_1$ and $x_2$, with

\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) = 0
\]

and we get the last equality since $c$ is in $I^o$ since $c$ is between $x_1$ and $x_2$, and they are in the interval $I^o$. So $f(x_2) - f(x_1) = 0$ and $f(x_2) = f(x_1)$. Since $x_1$ and $x_2$ were arbitrary members of $I$ with $x_1 < x_2$, this means $f$ is constant on $I$.

Case 2. Here we suppose $f'(x) > 0$ for all $x$ in $I^o$. Let $x_1, x_2$ be in $I$ with $x_1 < x_2$. (We aim to show $f(x_1) < f(x_2)$.) So the closed interval $[x_1, x_2]$ is contained in $I$, and the open interval $(x_1, x_2)$ is contained in $I^o$. So $f$ is continuous on $[x_1, x_2]$ and differentiable on $(x_1, x_2)$. So the Mean Value Theorem applies. So there is $c$ between $x_1$ and $x_2$, with

\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0
\]
and we get the last inequality since \( c \) is in \( I^o \) as above. Since \( x_2 - x_1 > 0 \), we multiply the inequality by \( x_2 - x_1 \), and we now have \( f(x_2) - f(x_1) > 0 \) and \( f(x_2) > f(x_1) \). Since \( x_1 \) and \( x_2 \) were arbitrary members of \( I \) with \( x_1 < x_2 \), this means \( f \) is increasing on \( I \).

**Case 3.** Here we suppose \( f'(x) < 0 \) for all \( x \) in \( I^o \). Let \( x_1, x_2 \) be in \( I \) with \( x_1 < x_2 \). (We aim to show \( f(x_1) > f(x_2) \).) So the closed interval \([x_1, x_2]\) is contained in \( I \), and the open interval \((x_1, x_2)\) is contained in \( I^o \). So \( f \) is continuous on \([x_1, x_2]\) and differentiable on \((x_1, x_2)\). So the Mean Value Theorem applies. So there is \( c \) between \( x_1 \) and \( x_2 \), with

\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) < 0
\]

and we get the last inequality since \( c \) is in \( I^o \) as above. Since \( x_2 - x_1 > 0 \), we multiply the inequality by \( x_2 - x_1 \), and we now have \( f(x_2) - f(x_1) < 0 \) and \( f(x_2) < f(x_1) \). Since \( x_1 \) and \( x_2 \) were arbitrary members of \( I \) with \( x_1 < x_2 \), this means \( f \) is decreasing on \( I \).

This ends the proof of the theorem. \( \square \)

**Remark.** If there are numbers \( a, b, c \), with \( a < c < b \) and such that \( f \) is increasing on \((a, c]\) and decreasing on \([c, b]\), then \( f(c) \) is a *local maximum value*! If there are numbers \( a, b, c \), with \( a < c < b \) and such that \( f \) is decreasing on \((a, c]\) and increasing on \([c, b]\), then \( f(c) \) is a *local minimum value*!

**Example 1.** Problem. \( f(x) = x^3 - 3x + 2 \). Find where \( f \) is increasing and decreasing. Find all local extrema. Sketch the graph.

\[
f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x + 1)(x - 1).
\]

So \( f'(x) = 0 \) iff \( x = -1 \) or \( 1 \).

If \( x < -1 \), then \( x - 1 < x + 1 < 0 \), and so \( f'(x) = 3(x + 1)(x - 1) > 0 \).

If \( -1 < x < 1 \), then \( x - 1 < 0 < x + 1 \), and so \( f'(x) = 3(x + 1)(x - 1) < 0 \).

If \( 1 < x \), then \( 0 < x - 1 < x + 1 \), and so \( f'(x) = 3(x + 1)(x - 1) > 0 \).

Now the Theorem tells us that

\( f \) is increasing on \((-\infty, -1]\),

\( f \) is decreasing on \([-1, 1]\), and

\( f \) is increasing on \([1, +\infty)\).

This latter information tells us that

\( f(-1) \) is a local maximum value, and

\( f(1) \) is a local minimum value.

\( f(-1) = -1 + 3 + 2 = 4 \) and \( f(1) = 1 - 3 + 2 = 0 \).
Example 2. If \( f(x) = \frac{1}{x} \), then \( f'(x) = \frac{-1}{x^2} < 0 \) for all \( x \neq 0 \). \( f \) is continuous on \( (-\infty, 0) \) and on \( (0, \infty) \). So by the Theorem, \( f \) is decreasing on \( (-\infty, 0) \) and on \( (0, \infty) \).

Warning. \( f(x) = \frac{1}{x} \) is NOT decreasing on \( (-\infty, 0) \cup (0, \infty) \) because, for example, \( f(-1) = -1 < 1 = f(1) \), but if it were decreasing on \( (-\infty, 0) \cup (0, \infty) \) we should instead have \( f(-1) > f(1) \).
You know that
\[
2x = \frac{d}{dx}(x^2) = \frac{d}{dx}(x^2 - 1) = \frac{d}{dx}(x^2 + 17) = \frac{d}{dx}(x^2 + \sqrt{2}) = \frac{d}{dx}(x^2 + C)
\]
where \(C\) is any constant. Is there any other sort of solution to
\[
\frac{dy}{dx} = 2x?
\]
The next Theorem says, “No, there is not.”

**Theorem 2.** Suppose the functions \(f\) and \(g\) are continuous on the interval \(I\). If \(f'(x) = g'(x)\) for all \(x\) in its interior \(I^0\), then there is a constant \(C\) so that
\[
g(x) = f(x) + C \quad \text{for all } x \in I.
\]

**Proof.** Suppose \(f\) and \(g\) are as in the hypotheses. Set \(h(x) = g(x) - f(x)\). Then \(h\) is continuous on \(I\), and \(h'(x) = g'(x) - f'(x) = f'(x) - f'(x) = 0\) for any \(x\) in \(I^0\). Now by (1) of Theorem 1 \(h\) is constant on \(I\); i.e., there is a constant \(C\) so that \(g(x) - f(x) = h(x) = C\) for all \(x\) in \(I\). That is, \(g(x) = f(x) + C\) for all \(x\) in \(I\). \(\square\)

**Definition 3.** \(F\) is an antiderivative of \(f\) on the interval \(I\) if \(F\) is continuous on \(I\) and
\[
F'(x) = f(x) \quad \text{for all } x \in I^0.
\]

**Proposition 4.** Any antiderivative of \(x^n\) equals
\[
\frac{x^{n+1}}{n+1} + C, \quad \text{for } n \neq -1
\]
for some constant \(C\).

**Proof.**
\[
\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}(n+1)x^{n+1-1} = x^n.
\]
Now apply Theorem 2. \(\square\)

**Example 3.** Any antiderivative of \(x^4\) equals \(\frac{x^5}{5} + C\) for some constant \(C\), and any antiderivative of \(x^{-4}\) equals \(\frac{x^{-3}}{-3} + C\) for some constant \(C\).

**Example 4.** Any antiderivative of \(\cos x\) equals \(\sin x + C\) for some constant \(C\), and any antiderivative of \(\sin x\) equals \(-\cos x + C\) for some constant \(C\) because
\[
\frac{d}{dx}(-\cos x) = -- \sin x = \sin x.
\]

**Proposition 5.** If \(F\) and \(G\) are antiderivatives of \(f\) and \(g\), respectively, and \(c\) is a real number, then
- \(F + G\) is an antiderivative of \(f + g\), and
- \(cF\) is an antiderivative of \(cf\).
Proof.
\[
\frac{d}{dx}(F(x) + G(x)) = F'(x) + G'(x) = f(x) + g(x) \quad \text{and} \quad \frac{d}{dx}(cF(x)) = cF'(x) = cf(x).
\]

□

Example 5. Problem: Given \( f'(x) = 5x^4 + 6x^3 - 5x^2 + 11 \) and \( f(0) = -3 \), find \( f \).

Using the previous two Propositions (and Theorem 2), we see that

\[
f(x) = x^5 + \frac{6x^4}{4} - \frac{5x^3}{3} + 11x + C \quad \text{and} \quad -3 = f(0) = 0 + 0 - 0 + 0 + C = C
\]

and so

\[
f(x) = x^5 + \frac{3}{2}x^4 - \frac{5}{3}x^3 + 11x - 3.
\]

Now we use the above propositions to derive some laws of physics about a freely falling body near the earth’s surface.

In what follows \( a \) is acceleration, \( v \) is velocity, and \( p \) is position.

We start with the understanding that a freely falling body near the earth’s surface experiences a constant acceleration of 32 feet per second per second. That is,

\[
a(t) = -32, \quad \text{but} \quad v'(t) = a(t).
\]

So

\[
v(t) = -32t + C.
\]

Since \( v(0) = 0 + C = C \), we replace \( C \) by the constant \( v_0 \) (which denotes initial velocity.) That is,

\[
v(t) = -32t + v_0, \quad \text{but} \quad p'(t) = v(t).
\]

So

\[
p(t) = -16t^2 + v_0t + D.
\]

Since \( p(0) = 0 + D = D \), we replace \( D \) by the constant \( p_0 \) (which denotes initial position.) That is,

\[
p(t) = -16t^2 + v_0t + p_0
\]

where \( v_0 \) is the velocity at time 0 and \( p_0 \) is the position at time 0.