Members of random closed sets

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Abstract. The members of Martin-Löf random closed sets under a distribution studied by Barmpalias et al. are exactly the infinite paths through Martin-Löf random Galton-Watson trees with survival parameter $\frac{2}{3}$. To be such a member, a sufficient condition is to have effective Hausdorff dimension strictly greater than $\gamma = \log_2 \frac{3}{2}$, and a necessary condition is to have effective Hausdorff dimension greater than or equal to γ.

Keywords: random closed sets, computability theory.

1 Introduction

Classical probability theory studies intersection probabilities for random sets. A random set will intersect a given deterministic set if the given set is large, in some sense. Here we study a computable analogue: the question of which real numbers are “large” in the sense that they belong to some Martin-Löf random closed set.

Barmpalias et al. [2] introduced algorithmic randomness for closed sets. Subsequently Kjos-Hanssen [6] used algorithmically random Galton-Watson trees to obtain results on infinite subsets of random sets of integers. Here we show that the distributions studied by Barmpalias et al. and by Galton and Watson are actually equivalent, not just classically but in an effective sense.

For $0 \leq \gamma < 1$, let us say that a real x is a $\textsc{Member}_\gamma$ if x belongs to some Martin-Löf (ML-) random closed set according to the Galton-Watson distribution (defined below) with survival parameter $p = 2^{-\gamma}$. We show that for $p = \frac{2}{3}$, this is equivalent to x being a member of a Martin-Löf random closed set according to the distribution considered by Barmpalias et al.

In light of this equivalence, we may state that (i) Barmpalias et al. showed that in effect not every $\textsc{Member}_\gamma$ is ML-random, and (ii) Joe Miller and Antonio Montálban showed that every ML-random real is a $\textsc{Member}_\gamma$; the proof of their result is given in the paper of Barmpalias et al. [2] The way to sharpen these results go via *effective Hausdorff dimension.* Each ML-random real has effective Hausdorff dimension equal to one. In Section 3 we show that (i′) a $\textsc{Member}_\gamma$ may have effective Hausdorff dimension strictly less than one, and (ii′) every real of sufficiently large effective Hausdorff dimension (where some numbers strictly less than one are "sufficiently large") is a $\textsc{Member}_\gamma$.

2 Equivalence of two models

We write $\Omega = 2^{<\omega}$, and 2^ω, for the sets of finite and infinite strings over $2 = \{0,1\}$, respectively. If $\sigma \in \Omega$ is an initial substring (a prefix) of $\tau \in \Omega$ we write $\sigma \preceq \tau$; similarly $\sigma \prec x$ means that the finite string σ is a prefix of the infinite string $x \in 2^\omega$. The length of σ is $|\sigma|$. We use the standard notation $[\sigma] = \{x : \sigma \prec x\}$, and for a set $U \subseteq \Omega$, $[U]^{\preceq} := \bigcup_{\sigma \in U} [\sigma]$. Let $\mathcal{P}$ denote the power set operation. Following Kjos-Hanssen [6], for a real number $0 \le \gamma < 1$ (so $\frac{1}{2} < 2^{-\gamma} \le 1$), let $\lambda_{1,\gamma}$ be the distribution with sample space $\mathcal{P}(\Omega)$ such that each string in Ω has probability $2^{-\gamma}$ of belonging to the random set, independently of any other string. Let λ^*_γ be defined analogously, except that now

$$\lambda^*_\gamma(\{S : S \cap \{\sigma 0, \sigma 1\} = J\} = \begin{cases} 1-p & \text{if } J = \{\sigma 0\} \text{ or } J = \{\sigma 1\}, \text{ and} \\ 2p-1 & \text{if } J = \{\sigma 0, \sigma 1\}, \end{cases}$$

independently for distinct σ, for $p = 2^{-\gamma}$. [1] For $S \subseteq \Omega$, Γ_S, the closed set determined by S, is the (possibly empty) set of infinite paths through the part of S that is downward closed under prefixes:

$$\Gamma_S = \{x \in 2^\omega : (\forall \sigma \prec x)\, \sigma \in S\}.$$

The *Galton-Watson (GW) distribution for survival parameter* $2^{-\gamma}$, also known as the $(1,\gamma)$-induced distribution [6], and as the distribution of a *percolation limit set* [12], is a distribution $\mathbb{P}_{1,\gamma}$ on the set of all closed subsets of 2^ω defined by

$$\mathbb{P}_{1,\gamma}(E) = \lambda_{1,\gamma}\{S : \Gamma_S \in E\}.$$

Thus, the probability of a property E of a closed subset of 2^ω is the probability according to $\lambda_{1,\gamma}$ that a random subset of Ω determines a tree whose set of infinite paths has property E. Similarly, let

$$\mathbb{P}^*_\gamma(E) = \lambda^*_\gamma\{S : \Gamma_S \in E\}.$$

A Σ^0_1 *subset of* $\mathcal{P}(\Omega)$ is the image of a Σ^0_1 subset of $\mathcal{P}(\omega) = 2^\omega$ via an effective isomorphism between Ω and ω.

$S \in \mathcal{P}(\Omega)$ is called $\lambda_{1,\gamma}$-*ML-random* if for each uniformly Σ^0_1 sequence $\{U_n\}_{n\in\omega}$ of subsets of $\mathcal{P}(\Omega)$ with $\lambda_{1,\gamma}(U_n) \le 2^{-n}$, we have $S \notin \bigcap_n U_n$. In this case Γ_S is called $\mathbb{P}_{1,\gamma}$-ML-random. Similarly, $S \in \mathcal{P}(\Omega)$ is called λ^*_γ-*ML-random* if for each uniformly Σ^0_1 sequence $\{U_n\}_{n\in\omega}$ of subsets of $\mathcal{P}(\Omega)$ with $\lambda^*_\gamma(U_n) \le 2^{-n}$, we have $S \notin \bigcap_n U_n$. In this case Γ_S is called $\mathbb{P}^*_\gamma$-ML-random.

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[1] The notation $\lambda_{1,\gamma}$ is consistent with earlier usage [6] and is also easy to distinguish visually from λ^*_γ.

Lemma 1 (Axon [1]**).** *For* $2^{-\gamma}=\frac{2}{3}$, $\Gamma\subseteq 2^\omega$ *is* $\mathbb{P}^*_\gamma$*-ML-random if and only if* Γ *is a Martin-Löf random closed set under the distribution studied by Barmpalias et al.*

Thinking of S as a random variable, define further random variables

$$G_n=\{\sigma:|\sigma|=n\ \ \&\ \ (\forall\tau\preceq\sigma)\ \tau\in S\}$$

and $G=\bigcup_{n=0}^\infty G_n$. We refer to a value of G as a *GW-tree* when G is considered a value of the random variable under the $\lambda_{1,\gamma}$ distribution. (A *BBCDW-tree* is a particular value of the random variable analogous to G, for the distribution λ^*_γ.) We have $G\subseteq S$ and $\Gamma_G=\Gamma_S$. The set G may have "dead ends", so let

$$G_\infty=\{\sigma\in G:G\cap[\sigma]\text{ is infinite}\}.$$

Thus $G_\infty\subseteq G\subseteq S$, and values of G_∞ are in one-to-one correspondence with values of Γ_S.

Let e be the extinction probability of a GW-tree with parameter $p=2^{-\gamma}$,

$$e=\mathbb{P}_{1,\gamma}(\varnothing)=\lambda_{1,\gamma}(\{S:\Gamma_S=\varnothing\}).$$

For any number a let $\overline{a}=1-a$.

Lemma 2.

$$e=\overline{p}/p.$$

Proof. Notice that we are not assuming $\langle\rangle\in S$. We have $e=\overline{p}+pe^2$, because there are two ways extinction can happen: (1) $\langle\rangle\notin S$, and (2) $\langle\rangle\in S$ but both immediate extension trees go extinct.

We use standard notation for conditional probability,

$$\mathbb{P}(E\mid F)=\frac{\mathbb{P}(E\cap F)}{\mathbb{P}(F)};$$

in measure notation we may also write $\lambda(E\mid F)=\lambda(E\cap F)/\lambda(F)$.

Lemma 3. *For all* $J\subseteq\{\langle 0\rangle,\langle 1\rangle\}$,

$$\lambda_{1,\gamma}\left\{G_\infty\cap\{\langle 0\rangle,\langle 1\rangle\}=J\mid G_\infty\neq\varnothing\right\}=\lambda^*_\gamma[G_1=J].$$

Proof. By definition, $\lambda^*_\gamma[G_1=J]$ equals

$$(2p-1)\cdot\mathbf{1}_{J=\{\langle 0\rangle,\langle 1\rangle\}}+\sum_{i=0}^{1}(1-p)\cdot\mathbf{1}_{J=\{\langle i\rangle\}},$$

so we only need to calculate $\lambda_{1,\gamma}\left\{G_\infty\cap\{\langle 0\rangle,\langle 1\rangle\}=J\mid G_\infty\neq\varnothing\right\}$. By symmetry, and because the probability that $G_1=\varnothing$ is 0, it suffices to calculate this probability for $J=\{\langle 0\rangle,\langle 1\rangle\}$. Now if $G_1=\{\langle 0\rangle,\langle 1\rangle\}$ then $\langle\rangle$ survives and both immediate extensions are non-extinct. Thus the conditional probability that $G_1=\{\langle 0\rangle,\langle 1\rangle\}$ is $\frac{p(1-e)^2}{1-e}=p(1-e)$. By Lemma 2, this is equal to $2p-1$.

Lemma 4. *Let the number* $\mathfrak{p}_s$ *be defined by*

$$\mathfrak{p}_s = \lambda_{1,\gamma}(\langle j\rangle \in G \mid (G \cap (\langle j\rangle^\frown \Omega))_\infty = \varnothing \ \ \& \ \ \langle\rangle \in G)$$

for $j = 0$ *(or* $j = 1$*, which gives the same result). Let*

$$\lambda_f(\cdot) = \lambda_{1,\gamma}(\cdot \mid G_\infty = \varnothing \ \ \& \ \ \langle\rangle \in G).$$

Then $\lambda_f(\langle i\rangle \in G_1) = \mathfrak{p}_s$.

Proof. We have $\mathfrak{p}_s = p^2e^2/(pe) = pe = 1 - p$. Next, $\lambda_f[G_1 = \varnothing] = \frac{p(1-p)^2}{pe^2} = p^2$ and $\lambda_f[G_1 = \{\langle 0\rangle, \langle 1\rangle\}] = \frac{p^3e^4}{pe^2} = (\overline{p})^2$. Hence

$$\lambda_f[G_1 = J] = (1-p)^2 \cdot \mathbf{1}_{J=\{\langle 0\rangle,\langle 1\rangle\}} + \sum_{i=0}^{1} p(1-p) \cdot \mathbf{1}_{J=\{\langle i\rangle\}} + p^2 \cdot \mathbf{1}_{J=\varnothing},$$

and so $\lambda_f(\langle i\rangle \in G_1) = (1-p)^2 + p(1-p) = \mathfrak{p}_s$, as desired.

Let $\lambda_c = \lambda_{1,\gamma}(\cdot \mid G_\infty \neq \varnothing)$ be $\lambda_{1,\gamma}$ conditioned on $G_\infty \neq \varnothing$, and let λ_i be the distribution of $G_\infty \in \mathcal{P}(\Omega)$ conditional on $G_\infty \neq \varnothing$. Let μ_i, μ_f, μ_c be the distribution of the tree G corresponding to the set S under λ_i, λ_f, λ_c, respectively (so $\mu_i = \lambda_i$). We define a $\mu_i \times \mu_f \to \mu_c$ measure-preserving map $\psi : 2^\Omega \times 2^\Omega \to 2^\Omega$. The idea is to overlay two sets S_i, S_f, so that S_i specifies G_∞, and S_f specifies $G \backslash G_\infty$. Let $\psi(S_i, S_f) = G_i \cup S_f$ where G_i is the tree determined by S_i. By Lemma 4, this gives the correct probability for string $\sigma \notin G_\infty$ that is the neighbor of a string in G_∞ to be in G. By considering two cases (a string in G is in G_∞ or not) we can derive that ψ is measure-preserving.

Intuitively, a λ_i-ML-random tree may by van Lambalgen's theorem be extended to a λ_c-ML-random tree by "adding finite pieces randomly". To be precise, van Lambalgen's theorem holds in the unit interval $[0,1]$ with Lebesgue measure λ, or equivalently the space 2^ω. If (X,μ) is a measure space then using the measure-preserving map $\varphi : (X,\mu) \to ([0,1],\lambda)$ induced from the Carathéodory measure algebra isomorphism theorem [7], we may apply van Lambalgen as desired, and obtain

Theorem 1. *For each ML-random BBCDW-tree* H *there is a ML-random GW-tree* G *with* $G_\infty = H_\infty$.

We next prove that the live part of every infinite ML-random GW-tree is an ML-random BBCDW-tree.

Theorem 2. *For each* S*, if* S *is* $\lambda_{1,\gamma}$*-ML-random then* G_∞ *is* λ^*_γ*-random.*

Proof. Suppose $\{U_n\}_{n\in\omega}$ is a λ^*_γ-ML-test with $G_\infty \in \bigcap_n U_n$. Let $\Upsilon_n = \{S : G_\infty \in U_n\}$. By Lemma 3, $\lambda_{1,\gamma}(\Upsilon_n) = \lambda^*_\gamma(U_n)$. Unfortunately, Υ_n is not a Σ^0_1 class, but we can approximate it. While we cannot know if a tree will end up being infinite, we can make a guess that will usually be correct.

Let e be the probability of extinction for a GW-tree. By Lemma 2 we have $e = \frac{\overline{p}}{p}$, so since $p > 1/2$, $e < 1$. Thus there is a computable function $(n, \ell) \mapsto m_{n,\ell}$ such that for all n and ℓ, $m = m_{n,\ell}$ is so large that $e^m \leq 2^{-n}2^{-2\ell}$. Let Φ be a Turing reduction so that $\Phi^G(n, \ell)$, if defined, is the least L such that all the 2^ℓ strings of length ℓ either are not on G, or have no descendants on G at level L, or have at least $m_{n,\ell}$ many such descendants. Let

$$W_n = \{S : \text{ for some } \ell,\, \Phi^G(n, \ell) \text{ is undefined}\}.$$

Let $A_G(\ell) = G_\infty \cap \{0,1\}^{\leq \ell}$ be G_∞ up to level ℓ. Let the approximation $A_G(\ell, L)$ to $A_G(\ell)$ consist of the nodes of G at level ℓ that have descendants at level L. Let

$$V_n = \{S : A_G(\ell, L) \in U_n \text{ for some } \ell, \text{ where } L = \Phi^G(n, \ell)\}, \text{ and}$$

$$X_n = \{S : \text{ for some } \ell,\, L = \Phi^G(n, \ell) \text{ is defined and } A_G(\ell, L) \neq A_G(\ell)\}.$$

Note that $\Upsilon_n = \{S : \text{for some } \ell,\, A_G(\ell) \in U_n\ \}$, hence $\Upsilon_n \subseteq W_n \cup X_n \cup V_n$. Thus it suffices to show that $\cap_n V_n$, W_n, $\cap_n X_n$ are all $\lambda_{1,\gamma}$-ML-null sets.

Lemma 5. *$\lambda_{1,\gamma}(W_n) = 0$.*

Proof. If $\Phi(\ell)$ is undefined then there is no L, which means that for the fixed set of strings on G at level ℓ, they do not all either die out or reach m many extensions. But eventually this must happen, so L must exist.

Indeed, fix any string σ on G at level ℓ. Let k be the largest number of descendants that σ has at infinitely many levels $L > \ell$. If $k > 0$ then with probability 1, above each level there is another level where actually $k + 1$ many descendants are achieved. So we conclude that either $k = 0$ or k does not exist.

From basic computability theory, W_n is a Σ^0_2 class. Hence each W_n is a Martin-Löf null set.

Lemma 6. *$\lambda_{1,\gamma}(X_n) \leq 2^{-n}$.*

Proof. Let E_σ denote the event that all extensions of σ on level L are *dead*, i.e. not in G_∞. Let F_σ denote the event that σ has at least m many descendants on $G(L)$.

If $A_G(\ell, L) \neq A_G(\ell)$ then some $\sigma \in \{0,1\}^\ell \cap G$ has at least m many descendants at level L, all of which are dead. If a node σ has at least m descendants, then the chance that all of these are dead, given that they are on G at level L, is at most e^m (the eventual extinction of one is independent of that of another), hence writing $\mathbb{P} = \lambda_{1,\gamma}$, we have

$$\mathbb{P}(A_G(\ell, L) \neq A_G(\ell)) \leq \sum_{\sigma \in \{0,1\}^\ell} \mathbb{P}\{E_\sigma \ \&\ F_\sigma\} = \sum_{\sigma \in \{0,1\}^\ell} \mathbb{P}\{E_\sigma \mid F_\sigma\} \cdot \mathbb{P}\{F_\sigma\}$$

$$\leq \sum_{\sigma \in \{0,1\}^\ell} \mathbb{P}\{E_\sigma \mid F_\sigma\} \leq \sum_{\sigma \in \{0,1\}^\ell} e^m \leq \sum_{\sigma \in \{0,1\}^\ell} 2^{-n}2^{-2\ell} = 2^{-n}2^{-\ell}.$$

and hence

$$\mathbb{P}X_n \leq \sum_\ell \mathbb{P}\{A_G(\ell, L) \neq A_G(\ell)\} \leq \sum_\ell 2^{-n}2^{-\ell} = 2^{-n}.$$

X_n is Σ^0_1 since when L is defined, $A_G(\ell)$ is contained in $A_G(\ell, L)$, and $A_G(\ell)$ is Π^0_1 in G, which means that if the containment is proper then we can eventually enumerate (observe) this fact. Thus $\cap_n X_n$ is a $\lambda_{1,\gamma}$-ML-null set.

V_n is clearly Σ^0_1. Moreover $V_n \subseteq \Upsilon_n \cup X_n$, so $\lambda_{1,\gamma}(V_n) \leq 2 \cdot 2^{-n}$, hence $\cap_n V_n$ is a $\lambda_{1,\gamma}$-ML-null set.

3 Towards a characterization of members of random closed sets

For a real number $0 \leq \gamma \leq 1$, the γ-weight $\mathrm{wt}_\gamma(C)$ of a set of strings $C \subseteq \Omega$ is defined by

$$\mathrm{wt}_\gamma(C) = \sum_{w \in C} 2^{-|w|\gamma}.$$

We define several notions of randomness of individual reals. A *Martin-Löf (ML-)γ-test* is a uniformly Σ^0_1 sequence $(U_n)_{n<\omega}$, $U_n \subseteq \Omega$, such that for all n, $\mathrm{wt}_\gamma(U_n) \leq 2^{-n}$. A *strong ML-$\gamma$-test* is a uniformly Σ^0_1 sequence $(U_n)_{n<\omega}$ such that for each n and each prefix-free set of strings $V_n \subseteq U_n$, $\mathrm{wt}_\gamma(V_n) \leq 2^{-n}$. A real is (strongly) γ-random if it does not belong to $\cap_n [U_n]^{\preceq}$ for any (strong) ML-γ-test $(U_n)_{n<\omega}$. If $\gamma = 1$ we simply say that the real, or the set of integers $\{n : x(n) = 1\}$, is *Martin-Löf random (ML-random)*. For $\gamma = 1$, strength makes no difference. For a measure μ and a real x, we say that x is *Hippocrates μ-random* if for each sequence $(U_n)_{n<\omega}$ that is uniformly Σ^0_1, and where $\mu[U_n]^{\preceq} \leq 2^{-n}$ for all n, we have $x \notin \cap_n [U_n]^{\preceq}$. Let the ultrametric υ on 2^ω be defined by $\upsilon(x, y) = 2^{-\min\{n : x(n) \neq y(n)\}}$. The γ-energy [12] of a measure μ is

$$I_\gamma(\mu) := \iint \frac{d\mu(b) d\mu(a)}{\upsilon(a, b)^\gamma}.$$

x is *Hippocrates γ-energy random* if x is Hippocrates μ-random with respect to some probability measure μ such that $I_\gamma(\mu) < \infty$.

For background on γ-energy and related concepts the reader may consult the monographs of Falconer [3] and Mattila [11] or the on-line lecture notes of Mörters and Peres [12]. The terminology *Hippocrates random* is supposed to remind us of Hippocrates, who did not consult the oracle at Delphi, but instead looked for natural causes. An almost sure property is more effective if it is possessed by all Hippocrates μ-random reals rather than merely all μ-random reals. In this sense Hippocratic μ-randomness tests are more desirable than arbitrary μ-randomness tests.

Effective Hausdorff dimension was introduced by Lutz [8] and is a notion of partial randomness. For example, if the sequence $x_0 x_1 x_2 \cdots$ is ML-random, then the sequence $x_0 0 x_1 0 x_2 0 \cdots$ has effective Hausdorff dimension equal to $\frac{1}{2}$.

Let $\dim^1_H x$ denote the effective (or constructive) Hausdorff dimension of x; then we have $\dim^1_H(x) = \sup\{\gamma : x \text{ is } \gamma\text{-random}\}$ (Reimann and Stephan [14]).

Examples of measures of finite γ-energy may be obtained from the fact that if $\dim^1_H(x) > \gamma$ then x is Hippocrates γ-energy random [6]. If x is strongly γ-random then x is γ-random and so $\dim^1_H(x) \geq \gamma$.

Theorem 3 **([6]).** *Each Hippocrates γ-energy random real is a* $\textsc{Member}_\gamma$.

Here we show a partial converse:

Theorem 4. *Each* $\textsc{Member}_\gamma$ *is strongly γ-random.*

Proof. Let $\mathbb{P} = \lambda_{1,\gamma}$ and $p = 2^{-\gamma} \in (\frac{1}{2}, 1]$. Let $i < 2$ and $\sigma \in \Omega$. The probability that the concatenation $\sigma i \in G$ given that $\sigma \in G$ is by definition

$$\mathbb{P}\{\sigma i \in G \mid \sigma \in G\} = p.$$

Hence the absolute probability that σ survives is

$$\mathbb{P}\{\sigma \in G\} = p^{|\sigma|} = \left(2^{-\gamma}\right)^{|\sigma|} = \left(2^{-|\sigma|}\right)^{\gamma}.$$

Let U be any strong γ-test, i.e. a uniformly Σ^0_1 sequence $U_n = \{\sigma_{n,i} : i < \omega\}$, such that for all prefix-free subsets $U'_n = \{\sigma'_{n,i} : i < \omega\}$ of U_n, $\mathrm{wt}_\gamma(U'_n) \leq 2^{-n}$. Let U'_n be the set of all strings σ in U_n such that no prefix of σ is in U_n. Clearly, U'_n is prefix-free. Let

$$[V_n]^{\preceq} := \{S : \exists i \, \sigma_{n,i} \in G\} \subseteq \{S : \exists i \, \sigma'_{n,i} \in G\}.$$

Clearly $[V_n]^{\preceq}$ is uniformly Σ^0_1. To prove the inclusion: Suppose G contains some $\sigma_{n,i}$. Since G is a tree, it contains the shortest prefix of $\sigma_{n,i}$ that is in U_n, and this string is in U'_n. Now

$$\mathbb{P}[V_n]^{\preceq} \leq \sum_{i \in \omega} \mathbb{P}\{\sigma'_{n,i} \in G\} = \sum_{i \in \omega} 2^{-|\sigma'_{n,i}|\gamma} \leq 2^{-n}.$$

Thus V is a test for $\lambda_{1,\gamma}$-ML-randomness. Suppose x is a $\textsc{Member}_\gamma$. Let S be any $\lambda_{1,\gamma}$-ML-random set with $x \in \Gamma_S$. Then $S \notin \cap_n [V_n]^{\preceq}$, and so for some n, $\Gamma \cap [U_n]^{\preceq} = \varnothing$. Hence $x \notin [U_n]^{\preceq}$. As U was an arbitrary strong γ-test, this shows that x is strongly γ-random.

Corollary 1. *Let $x \in 2^\omega$. We have the implications*

$$dim^1_H(x) > \gamma \;\Rightarrow\; x \textit{ is a } \textsc{Member}_\gamma \;\Rightarrow\; dim^1_H(x) \geq \gamma.$$

Proof. Each real x with $\dim^1_H(x) > \gamma$ is β-capacitable for some $\beta > \gamma$ [13]. This implies that x is γ-energy random [6, Lemma 2.5] and in particular x is Hippocrates γ-energy random. This gives the first implication. For the second implication, we use the fact that each strongly γ-random real x satisfies $\dim^1_H(x) \geq \gamma$ (see e.g. Reimann and Stephan [14]).

The second implication of Corollary 1 does not reverse, as not every real with $\dim^1_H(x) \geq \gamma$ is strongly γ-random [14].

The first implication of Corollary 1 fails to reverse as well:

Proposition 1. *Let $0 < \gamma < 1$. There is a γ-energy random real of effective Hausdorff dimension exactly γ.*

Proof. Consider the probability measure μ on 2^ω such that $\mu([\sigma^\frown 0]) = \mu([\sigma^\frown 1])$ for all σ of even length, and such that $\mu([\sigma^\frown 0]) = \mu([\sigma])$ for each σ of odd length $f(k) = 2k + 1$. A computation shows that $I_\gamma(\mu) = \sum_{k=0}^\infty 2^{(2k+1)\gamma} 2^{-k}$ which is finite if and only if $\gamma < 1/2$. We find that μ-almost all reals are μ-random and have effective Hausdorff dimension exactly 1/2. By modifying $f(k)$ slightly we can get $I_\gamma(\mu) < \infty$ for $\gamma = 1/2$ while keeping the effective Hausdorff dimension of μ-almost all reals equal to 1/2. Namely, what is needed is that $\sum_{k=0}^\infty 2^{f(k)\gamma} 2^{-k} < \infty$. This holds if $\gamma = 1/2$ and $f(k) = 2k - 2(1-\varepsilon)\log k$ for any $\varepsilon > 0$ since $\sum_k k^{-(1+\varepsilon)} < \infty$. Since this $f(k)$ is asymptotically larger than $(2-\delta)k$ for any $\delta > 0$, the μ-random reals still have effective Hausdorff dimension 1/2. The example generalizes from $\gamma = 1/2$ to an arbitrary $0 < \gamma < 1$.

Writing implication known to be strict as ⇒ and other implication as →, we have

> $\dim^1_H(x) > \gamma \Rightarrow x$ is γ-energy random $\rightarrow x$ is Hippocrates γ-energy random $\rightarrow x$ is a MEMBER$_\gamma$ $\rightarrow x$ is strongly γ-random $\Rightarrow x$ is γ-random $\Rightarrow \dim(x) \geq \gamma$.

By Reimann's effective capacitability theorem [13] x is strongly γ-random if and only if x is γ-capacitable.

Conjecture 1. There is a strongly γ-random real which is not Hippocrates γ-energy random.

Conjecture 2. A real x is a MEMBER$_\gamma$ if and only if x is Hippocrates γ-energy random.

To prove Conjecture 2, one might try to consult Lyons [10].

Proposition 2. *Let $0 < \gamma < 1$. If x is a real such that the function $n \mapsto x(n)$ is f-computably enumerable for some computable function f for which $\sum_{j<n} f(i) 2^{-n\gamma}$ goes effectively to zero, then x is not γ-random.*

Proof. Suppose $n \mapsto x(n)$ is f-c.e. for some such f, and let $F(n) = \sum_{j<n} f(n)$. Let α be any computable function such that $\alpha(n,i) \neq \alpha(n, i+1)$ for at most $f(n)$ many i for each n, and $\lim_{i\to\infty} \alpha(i,n) = x(n)$. Let $c(n,j)$ be the jth such i that is discovered for any $k < n$; so c is a partial recursive function whose domain is contained in $\{(n,j) : j \leq F(n)\}$. For a fixed i, α defines a real α_i by $\alpha_i(n) = \alpha(i,n)$. Let $V_n = \{x : \exists j \leq F(n)\ x \upharpoonright n = \alpha_{c(n,j)} \upharpoonright n)\}$. Since V_n is the union of at most $F(n)$ many cones $[x \upharpoonright n]$,

$$\mathrm{wt}_\gamma(V_n) \leq \sum_{j=1}^{F(n)} 2^{-n\gamma} = F(n) 2^{-n\gamma}$$

which goes effectively to zero by assumption. Thus there is a computable sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\mathrm{wt}_\gamma(V_{n_k}) \leq 2^{-k}$. Let $U_k = V_{n_k}$. Then U_k is Σ^0_1 uniformly in k, and $x \in \cap_k U_k$. Hence x is not γ-random.

Corollary 2 **([2]).** *No member of a ML-random closed set under the BBCDW distribution is f-c.e. for any polynomial-bounded f.*

Proof. If f is polynomially bounded then clearly $\sum_{j<n} f(i)2^{-n\gamma}$ goes effectively to zero. Therefore if x is f-c.e., x is not γ-random, hence not a $\textsc{Member}_\gamma$ for any $0 < \gamma < 1$, and thus not a member of a ML-random closed set under the BBCDW distribution.

Computing Brownian slow points. A function $f : \omega \to \omega$ is *diagonally nonrecursive* (DNR) if for each n, $f(n)$ is not equal to $\varphi_n(n)$, the value (if any) of the nth partial recursive function on input n. A real A is *Kurtz random relative to an oracle* B if it does not belong to any $\Pi^0_1(B)$ subset of 2^ω of fair-coin measure zero. Furthermore, B is Low(ML, Kurtz) if each real A that is ML-random is Kurtz random relative to B.

A starting point for the present paper was the observation (*) that each non-DNR Turing degree is Low(ML, Kurtz). A proof of this result due and credited to Kjos-Hanssen is given by Greenberg and Miller [4]; they prove that the converse holds as well. This can be used to show that each *slow point* (see Mörters and Peres [12]) of any ML-random Brownian motion must be of DNR Turing degree. The *fast* points on the other hand form a dense G_δ set, so there are fast points that are 1-generic and hence do not Turing compute any slow points.

In any case, the idea was initially to use the result (*) to understand members of random closed sets. However, as it turned out one could use the work of Hawkes [5] and Lyons [9] to better effect, in the present paper and in the precursor [6].

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