Numberings and randomness

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Abstract. We prove various results on effective numberings and Friedberg numberings of families related to algorithmic randomness. The family of all Martin-Löf random left-computably enumerable reals has a Friedberg numbering, as does the family of all Π^0_1 classes of positive measure. On the other hand, the Π^0_1 classes contained in the Martin-Löf random reals do not even have an effective numbering, nor do the left-c.e. reals satisfying a fixed randomness constant. For Π^0_1 classes contained in the class of reals satisfying a fixed randomness constant, we prove that at least an effective numbering exists.

1 Introduction

The general theory of numberings was initiated in the mid-1950s by Kolmogorov, and continued by Mal'tsev and Ershov [2]. A *numbering*, or *enumeration*, of a collection C of objects is a surjective map $F : \omega \to C$. In one of the earliest results, Friedberg [3, 1958] constructed an injective numbering ψ of the Σ^0_1 or computably enumerable (*c.e.*) sets such that the relation "$n \in \psi(e)$" is itself Σ^0_1. In a more general and informal sense, a numbering ψ of a collection of objects all having complexity $\mathcal{C}$ (such as n-c.e., Σ^0_n, or Π^0_n) is called *effective* if the relation "$x \in \psi(e)$" has complexity $\mathcal{C}$. If in addition the numbering is injective, then it is called a *Friedberg numbering*.

Brodhead and Cenzer [1] showed that there is an effective Friedberg numbering of the Π^0_1 classes in Cantor space 2^ω. They showed that effective numberings exist of the Π^0_1 classes that are homogeneous, and decidable, but not of the families consisting of Π^0_1 classes that are of measure zero, thin, perfect thin, small, very small, or nondecidable, respectively.

In this article we continue the study of existence of numberings and Friedberg numberings for subsets of ω and 2^ω. Many of our results are related to algorithmic randomness and in particular Martin-Löf randomness; see the books of Li and Vitányi [4] and Nies [6].

We now outline some notation and definitions used throughout. A subset T of $2^{<\omega}$ is a *tree* if it is closed under prefixes. The set $[T]$ of infinite paths through T is defined by $X \in [T] \leftrightarrow (\forall n) X \upharpoonright n \in T$, where $X \upharpoonright n$ denotes the initial segment $\langle X(0), X(1), \ldots, X(n-1)\rangle$. Next, P is a Π^0_1 class if $P = [T]$ for some computable tree T. Let $\sigma^\frown\tau$ denote the concatenation of σ with τ and let $\sigma^\frown i$ denote $\sigma^\frown\langle i\rangle$ for $i \in \omega$. The prefix ordering of strings is denoted by $\preceq$, so we have $\sigma \preceq \sigma^\frown\tau$. The string $\sigma \in T$ is a *dead end* if no extension $\sigma^\frown i$ is in T. For any $\sigma \in 2^{<\omega}$, $[\sigma]$ is the cone consisting of all infinite sequences extending σ. For a set of strings W, $[W]^{\preceq} = \bigcup_{\sigma\in W}[\sigma]$.

2 Families of left-c.e. reals

2.1 Basics

For our definition of left-c.e. reals we will follow the book of Nies [6]. Let $\mathbb{Q}_2$ be the set of dyadic rationals $\{\frac{a}{2^b} \leq 1 : a, b \in \omega\}$. For a dyadic rational q and real $x \in 2^\omega$, we say that $q < x$ if q is less than the real number $\sum_{i\in\omega} x(i)2^{-(i+1)}$.

Definition 1. *A real $x \in 2^\omega$ is* left-c.e. *if $\{q \in \mathbb{Q}_2 : q < x\}$ is c.e.*

Let $\leq_L$ denote lexicographic order on 2^ω. A dyadic rational may be written in the form $q = \sum_{i=1}^n a_i 2^{-i}$ where $a_n = 1$, and each $a_i \in \{0, 1\}$. The *associated binary string* of q is $s(q) = \langle a_1, \ldots, a_n\rangle$. (If $q = 0$ then $n = 0$ and the associated string is the empty string.) Conversely, the associated dyadic rational of $\sigma \in 2^{<\omega}$ is $\sum_{i=0}^{|\sigma|-1} \sigma(i)2^{-(i+1)}$.

Lemma 1. *For each $x \in 2^\omega$, we have that*

$$\{q \in \mathbb{Q}_2 : q < x\} \text{ is c.e.} \Leftrightarrow \{\sigma \in 2^{<\omega} : \sigma^\frown 0^\omega <_L x\} \text{ is c.e.}$$

Proof. We have that $\sigma^\frown 0^\omega <_L x$ iff the associated dyadic rational of σ is less than x, and $q < x$ iff the associated binary string σ of q satisfies $\sigma^\frown 0^\omega <_L x$. In fact, $\{s(q) : q \in \mathbb{Q}_2, q < x\} = \{\sigma : \sigma^\frown\omega <_L x\}$.

Definition 2. *An* effective numbering *of a family of left-c.e. reals $\mathcal{R}$ is an onto map $r : \omega \mapsto \mathcal{R}$ such that*

$$\{(q, e) \in \mathbb{Q}_2 \times \omega \mid q < r(e)\}$$

is c.e. If r is also injective then r is called a Friedberg numbering *of $\mathcal{R}$.*

Theorem 1. *The family of all left-c.e. reals has an effective numbering.*

Proof. Let $W_{e,s}$ be the e^{th} c.e. subset of $\mathbb{Q}_2$ as enumerated up to stage s. Let $r_{e,s}$ be the greatest element of $W_{e,s}$ and let $r(e) = \lim_{s\to\infty} r_{e,s}$. It is easy to check that r is an effective numbering of $\mathcal{R}$.

Some notions from algorithmic randomness will be needed repeatedly below. Ω is any fixed Martin-Löf random left-c.e. real with computable approximation $\Omega_s \leq_L \Omega_{s+1}$, $s \in \omega$. Let K denote prefix-free Kolmogorov complexity. Schnorr's Theorem states that a real $x \in 2^\omega$ is Martin-Löf random if and only if there is a constant c such that for all n, $K(x \upharpoonright n) \geq n - c$.

Theorem 2. *The family of all Martin-Löf random left-c.e. reals has an effective enumeration.*

Proof. Let K_t a uniformly computable approximation to Kolmogorov complexity at stage t, satisfying $K_{t+1} \leq K_t$. To obtain an enumeration of the Martin-Löf random left-c.e. reals, it suffices to enumerate all Martin-Löf random left-c.e. reals y such that $K(y \upharpoonright n) \geq n - c$ for all n, uniformly in c.

Initially our m^{th} ML-random left-c.e. real m_e will look like $r_e = r(e)$ from Theorem 1, i.e. $m_{e,s} = r_{e,s}$ unless otherwise stated. Let $r_{e,t}[n]$ be the associated string, restricted or appended with zeroes if necessary to obtain length n. If at some stage t, for some $n = n_t \in \omega$,

$$K_t(r_{e,t}[n]) < n - c,$$

then let $m_{e,s} = r_{e,t}[n]^\frown \Omega_s$ at all stages $s > t$ until, if ever, $K_s(r_{e,s}[n]) \geq n - c$ at some stage $s > t$. At this point, $r_{e,t}[n] < r_{e,s}[n]$, since r_e is a left-c.e. real. Resume where we left off in defining $m_e = r_e$, starting immediately at stage s with $m_{e,s} = r_{e,s}$. This process continues for the entire construction of each m_e.

This enumeration contains all left-c.e. reals which are Martin-Löf random with respect to the constant c, and only Martin-Löf left-c.e. random reals. Thus the merger of these enumerations over all c is an enumeration of all Martin-Löf random left-c.e. reals.

2.2 Kummer's method

Kummer [5, 1990] gave a priority-free proof of Friedberg's result. The conditions set forth in the proof provide a method of obtaining Friedberg numberings.

A *c.e. class* is a uniformly c.e. collection of subsets of ω (or equivalently, of $2^{<\omega}$ or $\mathbb{Q}_2$).

Theorem 3 (Kummer [5]). *If a c.e. class can be partitioned into two disjoint c.e. subclasses L_1 and L_2 such that L_1 is injectively enumerable and contains infinitely many extensions of every finite subset of any member of L_2, then the class is injectively enumerable.*

Theorem 4. *There is a Friedberg numbering of the left-c.e. reals.*

Proof. Let $C(x) = \{\tau : \tau^\frown 0^\omega <_L x\}$. Let

$$\mathcal{L} = \{C(x) : x \text{ is left-c.e.}\},$$

$$L_1 = \{C(x) : x(n) = 1 \text{ for an odd finite number of } n\},$$

and $L_2 = \mathcal{L} \setminus L_1$. It is clear that L_1 is injectively enumerable, and each finite subset F of a member of L_2 is contained in infinitely many members of L_1. The non-trivial part is to see that L_2 is c.e. Briefly, the idea is that we modify an enumeration $\{r_e\}_{e\in\omega}$ of all left-c.e. reals to only allow 1s to be added and removed in pairs of two. That is, we let $r^*_{e,s}$ be the longest prefix σ of the string associated with $r_{e,s}$ such that the number of 1s in σ is even. If in the end there are infinitely many 1s in r_e then $r^*_e = r_e$, and it is clear that $r^*_{e,s} \leq r^*_{e,s+1}$.

Theorem 5. *There is a Friedberg numbering of the Martin-Löf random left-c.e. reals.*

Proof. Let

$$\mathcal{R} = \{C(x) : x \text{ is ML-random and left-c.e.}\},$$

$L_1 = \{C(1^n{}^\frown\Omega) : n \in \omega\}$, and $L_2 = \mathcal{R}\backslash L_1$. Again, it is clear that L_1 is injectively enumerable and each finite subset of a member of L_2 can be extended to infinitely many members of L_1. We will argue that L_2 is c.e. Note that $1^n{}^\frown\Omega <_L 1^{n+1}{}^\frown\Omega$ for each n. Thus

$$L_2 = \bigcup_{n\in\omega} \left\{C(y) \in \mathcal{R} : 1^n{}^\frown\Omega <_L y <_L 1^{n+1}{}^\frown\Omega\right\}.$$

so it suffices to show that the sets

$$\{C(y) \in \mathcal{R} : y <_L 1^n{}^\frown\Omega\}, \tag{1}$$

$$\{C(y) \in \mathcal{R} : 1^n{}^\frown\Omega <_L y\} \tag{2}$$

are uniformly c.e.

Notice that $y <_L 1^n{}^\frown\Omega$ iff there is some k such that $y \upharpoonright k <_L (1^n{}^\frown\Omega) \upharpoonright k$, so for (1) it suffices to show that $\{C(y) \in \mathcal{R} : y \upharpoonright k <_L (1^n{}^\frown\Omega) \upharpoonright k\}$ is c.e., uniformly in n and k. This is non-trivial only if $k > n$, and in fact it suffices to show that a suitable subfamily $\mathcal{F}_k$ of $\{C(y) \in \mathcal{R} : y <_L \Omega\}$ containing

$$\{C(y) \in \mathcal{R} : y \upharpoonright k <_L \Omega \upharpoonright k\} \tag{1'}$$

is uniformly c.e. for $k \in \omega$.

We modify the enumeration $\{m_e\}_{e\in\omega}$ of the left-c.e. random reals from Theorem 2, producing a new enumeration $\{\widehat{m}_e\}_{e\in\omega}$. Initially, as long as $\Omega \upharpoonright k$ looks like the constant-zero string 0^k then $\widehat{m}_e$ is made to look like $0^\frown\Omega$. Note that since $\Omega \neq 0^\omega$, $0^\frown\Omega <_L \Omega$.

If at any stage it looks like $\Omega \upharpoonright k \neq 0^k$ then thereafter we let $\widehat{m}_e = m_e$ as long as $m_e \upharpoonright k <_L \Omega \upharpoonright k$. If at some stage s, $m_{e,s} \upharpoonright k \geq_L \Omega_s \upharpoonright k$, then we say that we are in an undesirable state, and we let $\widehat{m}_{e,t} = m_{e,s-1} \upharpoonright k^\frown\Omega_t$ for all $t \geq s$ until a possible later stage where we are in a desirable state again.

Thus, if m_e really satisfies $m_e \upharpoonright k <_L \Omega \upharpoonright k$ then we will have $\widehat{m}_e = m_e$, and if not then $\widehat{m}_e$ will be a finite string $\sigma <_L \Omega \upharpoonright k$ followed by Ω, so in any case it will be a Martin-Löf random real. Thus $\{C(\widehat{m}_e)\}_{e\in\omega}$ is an effective enumeration of a family $\mathcal{F}_k$ as stated. The argument for (2) is analogous.

2.3 Specifying randomness constants

Recall that Schnorr's Theorem states that a real $x \in 2^\omega$ is Martin-Löf random if and only if there is a constant c such that for all n, $K(x \upharpoonright n) \geq n - c$. The optimal randomness constant of x is the least c such that this holds. For each interval $I \subseteq \omega$ we let $\mathcal{A}_I$ $(\mathcal{R}_I)$ denote the set of all Martin-Löf random (and left-c.e., respectively) reals whose optimal randomness constant belongs to I. Let μ

denote the fair-coin Cantor-Lebesgue measure on 2^ω. By the proof of Schnorr's Theorem we have

$$\mu(\{x : (\forall n) K(x \restriction n) \geq n - c\}) \geq 1 - 2^{-(c+1)}.$$

Consequently, if $c \geq 0$, then $\mu\mathcal{A}_{[0,c]} > 0$ and $\mathcal{A}_{[0,c]} \neq \varnothing$.

Theorem 6. *Let $c \geq 0$. There is no effective enumeration of $\mathcal{R}_{[0,c]}$.*

Proof. Suppose that $\{\alpha_e\}_{e\in\omega}$ is such an enumeration, with a uniformly computable approximation $\alpha_{e,s}$ such that $\alpha_e = \lim_{s\to\infty} \alpha_{e,s}$ and $\alpha_{e,s} \leq \alpha_{e,s+1}$. Note that

$$\mathcal{A}_{[0,c]} = \{x : (\forall n) K(x \restriction n) \geq n - c\}$$

is a Π^0_1 class. Let $\beta_s = \max\{\alpha_{e,s} : e \leq s\}$. Then $\beta = \lim_{s\to\infty} \beta_s$ is left-c.e., and since the left-c.e. members of $A_{[0,c]}$ are dense in $A_{[0,c]}$, β is the rightmost path of $A_{[0,c]}$. However the rightmost path of a Π^0_1 class is also *right-c.e.*, defined in the obvious way. Thus β is a Martin-Löf random real that is computable, a contradiction.

Theorem 7. *For each c there is an effective numbering of $\mathcal{R}_{[c+1,\infty)}$.*

Proof sketch. Let $\{m_e\}_{e\in\omega}$ be an effective enumeration of all left-c.e. random reals, with the additional property that for each e there are infinitely many e' such that for all s, $m_{e,s} = m_{e',s}$. We will define an effective numbering $\{\alpha_e\}_{e\in\omega}$ of $\mathcal{R}_{[c+1,\infty)}$.

We say that a string σ *satisfies randomness constant c at stage t* if

$$K_t(\sigma) \geq |\sigma| - c;$$

otherwise, we say that σ fails randomness constant c at stage t.

We proceed in stages $t \in \omega$, monitoring each $m_{e,t}$ for $e \leq t$ at stage t. If for some t_0, n, e, we observe that $m_{e,t_0}[n]$ fails randomness constant c, then we want to assign a place for m_e in our enumeration of $\mathcal{R}_{[c+1,\infty)}$. So we let d be minimal so that α_d has not yet been mentioned in the construction, and let $\alpha_{d,s} = m_{e,s}$ for all stages $s \geq t_0$ until further notice. If $m_{e,t_1}[n]$ at some stage $t_1 \geq t_0$ satifies randomness constant c, then we *regret* having assigned m_e a place in our enumeration $\{\alpha_e\}_{e\in\omega}$. To compensate for this regret, we choose a large number $p = p_{c,n}$ and for all stages $s \geq t_1$ let $\alpha_{d,s} = m_{e,s}[n]^\frown 0^{p\frown} \Omega_s$. The largeness of p guarantees that $m_{e,s}[n]^\frown 0^p$ does not satisfy randomness constant c. [1] If m_e actually does fail randomness constant c, but at a larger length $n' > n$, then because there are infinitely many e' with $m_{e'} = m_e$ we will eventually assign some $\alpha_{d'}$ to some such $m_{e'}$ at a stage t_2 that is so large that $m_{e',t_2}[n'] = m_{e'}[n']$. Thus, each real in $\mathcal{R}_{[c+1,\infty)}$ will eventually be assigned a permanent $\alpha_{d'}$.

[1] To be precise, if $|\sigma| = n$ then there are universal constants $\hat{c}$ and $\tilde{c}$ such that, thinking of p sometimes as a string, $K(\sigma^\frown 0^p) \leq K(\sigma) + K(p) + \hat{c} \leq 2|\sigma| + 2|p| + \tilde{c} = 2n + 2\log p + \tilde{c} \leq n + p - c$ provided $p - 2\log p \geq n + \tilde{c} + c$, which is true for $p = p_{n,c}$ that we can find effectively.

Remark 1. We believe that one can even show that there is a Friedberg numbering of $\mathcal{R}_{[c+1,\infty)}$. The idea is to modify L_1 so that the strings 1^n are replaced by 1^{d_c+n} for a sufficiently large d_c, as in the footnote on page 5.

Remark 2. Theorems 6 and 7 indicate perhaps that the left-c.e. members of Σ^0_2 classes are generally easier to enumerate than those of Π^0_1 classes; this may be due to the "Σ^0_n nature" of left-c.e. reals (for $n = 1$).

Family	Enumeration?	Friedberg?
All Π^0_1 classes		Yes, by Theorem 10
All left-c.e. reals		Yes, by Theorem 4
Π^0_1 classes C, $\mu C > 0$		Yes, by Theorem 10
Left-c.e. reals in MLR		Yes, by Theorem 5
Π^0_1 classes $\subseteq \mathcal{A}_{[0,c]}$	Yes, by Proposition 2	Open problem
Π^0_1 classes $\subseteq$ MLR	No, by Theorem 8	
Left-c.e. reals in $\mathcal{A}_{[0,c]}$	No, by Theorem 6	

Fig. 1. Existence of effective numberings and Friedberg numberings, where MLR $= \bigcup_{c\in\omega} \mathcal{A}_{[0,c]}$.

Whether a set of the form $\mathcal{A}_{[c_1,c_2]}$ for $0 \le c_1 \le c_2 < \infty$ is nonempty appears to depend on the universal prefix machine on which Kolmogorov complexity is based.

Question 1. Does there exist $0 \le c_1 \le c_2 < \infty$ and a choice of universal machine underlying Kolmogorov complexity such that $\mathcal{A}_{[c_1,c_2]}$ has no effective enumeration?

3 Families of Π^0_1 classes

Definition 3 ([1]). *Let $\mathcal{C}$ be a family of closed subsets of 2^ω. We say that $\mathcal{C}$ has a* computable enumeration *if there is a uniformly computable collection $\{T_e\}_{e\in\omega}$ of trees $T_e \subseteq 2^{<\omega}$ (that is, $\{\langle\sigma, e\rangle : \sigma \in T_e\}$ is computable, and $\sigma^\frown\tau \in T_e$ implies $\sigma \in T_e$) such that $\mathcal{C} = \{[T_e] : e \in \omega\}$.*

Definition 4 ([1]). *Let $\mathcal{C}$ be a family of closed subsets of 2^ω. We say that $\mathcal{C}$ has an* effective enumeration *if there is a Π^0_1 set $S \subseteq 2^\omega \times \omega$, such that $\mathcal{C} = \{\{X : (X, e) \in S\} : e \in \omega\}$.*

Proposition 1. *Let $\mathcal{C}$ be a family of closed subsets of 2^ω. The following are equivalent:*

(1) $\mathcal{C}$ has a computable enumeration;
(2) $\mathcal{C}$ has an effective enumeration.

Proof. (1) implies (2): Let $\{T_e\}_{e\in\omega}$ be given, and define

$$S = \{(X,e) : \forall n\ X \upharpoonright n \in T_e\}.$$

(2) implies (1): Let S be given, let Φ_a be a Turing functional such that $(X,e) \in S \Leftrightarrow \Phi_a^X(e)\uparrow$, and let $T_e = \{\sigma \in 2^{<\omega} : \Phi_{a,|\sigma|}^{\sigma}(e)\uparrow\}$.

In light of Proposition 1, we may use either notion. Note that if C belongs to a family as in Proposition 1 then C is a Π_1^0 class.

3.1 Existence of numberings

Theorem 8. *Let $P \subseteq 2^\omega$, let $\mathcal{C}_P$ be the collection of all Π_1^0 classes contained in P, and let $\mathcal{N}_P$ be the collection of all nonempty Π_1^0 classes contained in P. Assume P has the following properties:*

(i) P is co-dense: no cone $[\sigma]$, $\sigma \in 2^{<\omega}$, is contained in P;
(ii) P is closed under shifts: if $x \in P$ then $\sigma^\frown x \in P$;
(iii) $\mathcal{N}_P \neq \emptyset$.

Then there is no effective numbering of either $\mathcal{C}_P$ or $\mathcal{N}_P$.

Proof. If there is a numbering of $\mathcal{N}_P$ then there is one of $\mathcal{C}_P$, because if $\emptyset \in \mathcal{C}_P$ (as is always the case) we may simply add an index of $\emptyset$ to the numbering. Thus it suffices to show that there is no effective numbering of $\mathcal{C}_P$. Suppose to the contrary that $e \mapsto [T_e]$ enumerates the family of Π_1^0 classes in $\mathcal{C}_P$. By (iii), we may assume $[T_0] \neq \emptyset$. By (i), T_0 has infinitely many dead ends. Let the dead ends of T_0 be listed in a computable way (for instance, by length-lexicographic order), as σ_n, $n \in \omega$. By (i) again, we may let τ_n be the least extension of σ_n which extends a dead end of T_n. Define a computable tree T by putting T_0 above τ_n. That is, let $[T] \cap [\tau_n] = [\tau_n T_0]$ and $[T] = [T_0] \cup \bigcup_n [\tau_n T_0]$. By (ii), the resulting class $[T]$ belongs to $\mathcal{C}_P$. Since $[T_0] \neq \varnothing$, $[T] \cap [\tau_n] \neq \emptyset = [T_n] \cap [\tau_n]$, so $[T]$ is not contained in or equal to any $[T_n]$.

All assumptions (i), (ii), (iii) of Theorem 8 are necessary: consider $P = 2^\omega$, $P = \{x\}$, where x is a single computable real, and $P = \emptyset$, respectively.

Corollary 1. *The following families of Π_1^0 classes have no effective numbering:*

1. *Π_1^0 classes containing only Martin-Löf random reals;*
2. *special Π_1^0 classes (those containing only non-computable reals);*
3. *Π_1^0 classes containing only reals x such that the Muchnik degree* [7] *of $\{x\}$ is above a fixed nonzero Muchnik degree;*
4. *Π_1^0 classes containing only finite (or only co-finite) subsets of ω.*

Proposition 2. *(1) The family of all Π_1^0 classes containing only reals that are Martin-Löf random with respect to a fixed randomness constant is effectively enumerable.*

(2) The family of all Σ^0_2 classes containing only Martin-Löf random reals is effectively enumerable.

Proof. (1). We enumerate all Π^0_1 classes as $\{P_i\}_{i\in\omega}$ and let

$$Q_i = P_i \cap \{x : \forall n\ K(x \upharpoonright n) \geq n - c\}.$$

Then $\{Q_i\}_{i\in\omega}$ is an enumeration of all Π^0_1 classes containing only reals that are Martin-Löf random with randomness constant c. Part (2) is analogous.

We may sum up the situation by stating that it is only the mixture of Π^0_1 and Σ^0_2 classes that leads to the negative result of Corollary 1(1). The proof of Theorem 8 for the case in Corollary 1(1) proves the following basic property of Martin-Löf tests.

Corollary 2. *For each Martin-Löf test $\{U_n\}_{n\in\omega}$ there is a Σ^0_1 class V containing all non-Martin-Löf random reals but containing no set U_n, $n \in \omega$.*

As is well-known, all Π^0_1 classes containing Martin-Löf random reals have positive measure. In contrast to Corollary 1(1), such classes can be effectively enumerated:

Theorem 9. *There is an effective numbering of the Π^0_1 classes of positive measure.*

Proof. It suffices to enumerate, uniformly in $n \in \omega$, all Π^0_1 classes of measure at least $r := \frac{1}{n}$. To accomplish this, let $e \mapsto W_e$ be an effective numbering of Σ^0_1 sets of strings, which gives rise to all Π^0_1 classes. That is, if P is a Π^0_1 class, then $P = 2^\omega \setminus [W_e]^{\preceq}$ for some e. Modify this enumeration so that strings enumerate into each W_e so long as the overall measure never surpasses $1 - r$. More precisely, if, at some stage $s > 0$, some σ is supposed to enter $W_{e,s}$ but this causes the measure of $[W_e]^{\preceq}$ to surpass $1 - r$, then we hereafter discontinue to enumerate strings into W_e; call this modified set $\widehat{W}_{e;n}$. It follows that $e \mapsto \widehat{W}_{e;n}$ is a numbering that gives rise to all Σ^0_1 classes of measure at most $1 - \frac{1}{n}$. Then the sequence of sets $\left\{\left[\widehat{W}_{e;n}\right]^{\preceq}\right\}$ for $\langle e, n\rangle \in \omega \times \omega$ is an effective enumeration of the Σ^0_1 classes of measure less than 1.

This contrasts with the result of [1] that there is no effective numbering of the Π^0_1 classes of measure zero. We next show that any effectively enumerable family of Π^0_1 classes containing all the clopen classes has a Friedberg numbering. In fact, we show something slightly stronger.

Definition 5. *The* optimal covering *of $S \subseteq 2^{<\omega}$ is*

$$O = O_S = \{\sigma : [\sigma] \subseteq [S]^{\preceq}\ \ \&\ \ \neg(\exists \tau \prec \sigma)([\tau] \subseteq [S]^{\preceq})\}.$$

Let $\mathfrak{A}$ be the family of all sets O that have odd cardinality and are optimal coverings of sets S.

Theorem 10. *Any effectively enumerable family of Σ^0_1 classes $\mathcal{F}$ with $\mathcal{F} \supseteq \{[O]^{\preceq} : O \in \mathfrak{A}\}$ has a Friedberg numbering.*

Proof. For a set $Z \subseteq 2^{<\omega}$, we say that Z is *filter closed* if Z is closed under extensions ($\sigma \in Z \Rightarrow \sigma^\frown\tau \in Z$) and such that whenever both $\sigma^\frown 0$ and $\sigma^\frown 1$ are in Z then $\sigma \in Z$. The *filter closure* of Y is the intersection of all filter closed sets containing Y and is denoted by $Y^{\uparrow}$.

Since $\mathcal{F}$ is effectively enumerable, we may let $e \mapsto Y_e$ be a numbering of all filter closed sets of strings with $[Y_e]^{\preceq} \in \mathcal{F}$. Since $Y_e \neq Y_{e'}$ implies $[Y_e]^{\preceq} \neq [Y_{e'}]^{\preceq}$, it suffices to injectively enumerate these sets Y_e. Let

$$L_1 = \left\{O^{\uparrow} : O \in \mathfrak{A}\right\} \text{ and } L_2 = \{Y_e : Y_e \notin L_1\}.$$

It is clear that L_1 is injectively enumerable. By the assumption of the theorem, each $[O^{\uparrow}]^{\preceq} \in \mathcal{F}$. It is also clear that each finite subset of any $Y \in L_2$ is contained in infinitely many $O^{\uparrow} \in L_1$.

We claim that L_2 has an effectively enumeration $\{Y_e^*\}_{e\in\omega}$, to be constructed below. Fix e and let $Y_e = \{\sigma_n\}_{n\in\omega}$ in order of enumeration.

$S \subseteq 2^{<\omega}$ is an *acceptable family* if its optimal covering O has finite even cardinality. In particular $O \notin \mathfrak{A}$. We say that stage n is *good* if σ_n has greater length than any member of $O_n = O_{S_n}$ for $S_n = \{\sigma_0, \ldots, \sigma_{n-1}\}$ and does not extend any member of O_n.

Construction. We will construct Y_e^* as $Y_e^* = \bigcup_{n\in\omega} Y_{e,n}$ for uniformly computable sets $Y_{e,n}$. We set $Y_{e,-1} = \varnothing$. Suppose $n \geq 0$. If stage n is not good, we keep $Y_{e,n} = Y_{e,n-1}$.

If stage n is good, there are two cases.

Case a. S_n is an acceptable family. Then let $Y_{e,n}$ be the filter closure of O_n.

Case b. Otherwise. Then let $Y_{e,n}$ be the filter closure of $O_n \cup \{\sigma_n\}$.

We separately enumerate all sets generated from any acceptable family whose optimal covering has finite even cardinality. (*)

End of Construction.

Verification. Note that in both Case a and Case b, $Y_{e,n}$ is the filter closure of an acceptable family, so we do not enumerate any member of L_1. By (*), it therefore suffices to show that we enumerate all sets generated from an infinite family, i.e. non-clopen sets, and that each Y_e^* is some $Y_{e'}$.

If Y_e is not clopen then there are infinitely many good stages. Then in the end $Y_e^* = Y_e$, because σ_n is covered either right away (case b) or at the next good stage (case b).

Corollary 3. *The family of all Σ^0_1 classes of measure less than one, or equivalently Π^0_1 classes of positive measure, has a Friedberg numbering.*

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