

The 4th problem in Complex Analysis Comp. Exam given in 2007

**Problem 1** Suppose that for all  $z \in D = \{z \in \mathbb{C} : |z| < 1\}$  and all  $n \in \mathbb{N} = \{0, 1, 2, \dots\}$  we have the  $f_n$  is holomorphic in  $D$  and  $|f_n(z)| < 1$ . Also suppose that  $\lim_{n \rightarrow 1} \operatorname{Im} f_n(x) = 0$  for all  $x \in (-1, 0)$ . Then  $\lim_{n \rightarrow \infty} \operatorname{Im} f_n(1/2) = 0$ .

**Proof.** Suppose

$$\limsup |\operatorname{Im} f_n(1/2)| = \alpha > 0$$

for some  $\alpha \in \mathbb{R}$ . Then  $\{f_n\}_n$  has a subsequence  $\{f_{n_j}\}_{n_j}$  such that

$$\lim \operatorname{Im} f_{n_j}(1/2) = \alpha \neq 0$$

or

$$\lim \operatorname{Im} f_{n_j}(1/2) = -\alpha \neq 0$$

W.L.O.G, suppose

$$\lim \operatorname{Im} f_{n_j}(1/2) = \alpha$$

we'll arrive at a contradiction. Since  $\{f_n\}_n$  is a normal family on  $D$ ,  $\{f_{n_j}\}_{n_j}$  has a subsequence that converges locally uniformly to some analytic function  $f$  defined on  $D$ . To make notations concise, let this subsequence be  $\{f_{n_j}\}_{n_j}$  itself.

(The following is completed with hints given by Dr. Smith that, use the Taylor series representation of  $f(z)$  and try to show if  $f$  is analytic in  $D$  that maps  $(-1, 0)$  to a subset of  $\mathbb{R}$ , then  $f$  must map  $(0, 1)$  also to a subset of  $\mathbb{R}$  in the general setting. Show that  $f^{(n)}(-\varepsilon)$  is real for any  $\varepsilon$  with  $1 > \varepsilon > 0$ ).

Now, take any  $x_0$  with  $-1 < x_0 < 0$ . Then

$$\operatorname{Im} f(x_0) = \lim \operatorname{Im} f_{n_j}(x_0) = 0$$

and  $f((-1, 0)) = \{0\}$ . So

$$f'(x_0) = \lim_{y \rightarrow x_0, y \in (-1, 0)} \frac{f(y) - f(x_0)}{y - x_0} = 0$$

i.e.,  $f'((-1, 0)) = \{0\}$ . Suppose when  $n = m$  for some  $m \in \mathbb{N}^+$ ,  $m \geq 2$

$$f^{(m)}((-1, 0)) = \{0\}$$

then

$$f^{(m+1)}(x_0) = \lim_{y \rightarrow x_0, y \in (-1, 0)} \frac{f^{(m)}(y) - f^{(m)}(x_0)}{y - x_0} = 0$$

that is,  $f^{(m+1)}((-1, 0)) = \{0\}$ . Thus the induction principle implies

$$f^{(n)}(x) \equiv 0$$

for all  $n \in \mathbb{N}$  and  $x \in (-1, 0)$ . Further the continuity of each  $f^{(n)}$ ,  $n \in \mathbb{N}$  in  $D$  implies

$$f^{(n)}(0) \equiv \lim_{k \rightarrow \infty, k \in \mathbb{N}^+} f^{(n)}\left(-\frac{1}{k}\right) = 0$$

Being analytic in  $D$ , it's clear that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \equiv 0$$

for all  $z \in D$ . Consequently,  $f \equiv 0$  and particularly  $f(1/2) = 0$ . But

$$0 = f(1/2) = \lim_{n_j \rightarrow \infty} f_{n_j}(1/2) = \alpha \neq 0$$

which is an obvious contradiction.

Whence,  $\lim_{n \rightarrow \infty} \operatorname{Im} f_n(1/2) = 0$  and the assertion is justified. ■