Function Space and Montel's Theorem

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January 19, 2008

Contents

Abstract

This theorem touches the final topic required by the comprehensive exam in complex.

1 Notations

$\mathbb{C}$ the complex plane. Ω a domain in $\mathbb{C}$. $\mathfrak{F}$ a family of functions f. (S, d) a metric space where f assumes value. $C(U)$ the set of all continuous function defined on the open set U. For sequence of functions $\Rightarrow$ means uniform convergence on the specified set. Δ always dontes closed disk

2 Arzelà-Ascoli Theorem and Montel's Theorem

Definition 2.1 *The functions in a family $\mathfrak{F}$ are said to be equicontinuous on a set $E \subseteq \Omega$ iff for each $\varepsilon > 0$,there exists a $\delta > 0$ such that $d(f(z), f(z_0)) < \varepsilon$ whenver $|z - z_0| < \delta$ and $z, z_0 \in E$,simultaneously for all functions $f \in \mathfrak{F}$.*

Definition 2.2 *A family $\mathfrak{F}$ is said to be normal in Ω if every sequence $\{f_n\}$ of functions $f_n \in \mathfrak{F}$ has a subsequence $\{f_{n_k}\}$which either converges uniformly or tends uniformly to ∞ on every compact subset of Ω.*

For the purpose of this note, the most significant feature of equicontinuity is that it bridges the gap between pointwise convergence and normal convergence.

Lemma 2.3 *Let (S,d) be a complete metric space, U be an open subset of S, K a compact subset of S contained in U.Then there is some $\rho > 0$ such that for all $z \in K$, $B(z,\rho) \subseteq U$.*

Proof. *The hypothesis implies that*

$$d(K, \partial U) = r > 0$$

else, $K \cap \partial U \neq \varnothing$ and $K \cap (S \sim U) \neq \varnothing$. To be more precise, suppose the contrary. Then for each $\rho_n = 1/n$ with $n \in \mathbb{N}$, there exists $z_n \in K$ such that $B(z_n, \rho_n) \cap (S \sim U) = \varnothing$.Since K is compact, (z_n) must have a subsequence $\left(z_{n_j}\right)$ such that

$$\lim_{j \to \infty} z_{n_j} = z_0 \in K \subseteq U$$

which further means there is some $\delta > 0$ such that $B(z_0, \delta) \subseteq U$.But for this $\delta > 0$,there exists $n_0 \in \mathbb{N}$ such that for all $n_j > n_0$,

$$z_{n_j} \in B(z_0, \delta)$$

and hence for each $n_j > n_0$, there is some $\delta_{n_j} > 0$ such that

$$B\left(z_{n_j}, \delta_{n_j}\right) \subseteq B(z_0, \delta)$$

Since $\lim_{n_j \to 1} \rho_{n_j} = 0$,then for sufficiently large n_j,it's clear that $\rho_{n_j} < \delta_{n_j}$ and

$$B(z_{n_j}, \rho_{n_j}) \subseteq B\left(z_{n_j}, \delta_{n_j}\right) \subseteq B(z_0, \delta)$$

which is a contradiction. ■

Lemma 2.4 *Let (f_n) be a sequence from an equicontinuous subfamily $\mathfrak{F}$ of $C(U)$.Suppose that this sequence converges pointwise in U.Then it converges normally in U.*

Proof. (Contrapositive) Let f be the pointwise limit function of (f_n) and K be any arbitrary compact subset of U.It suffices to show that,

(i) for any $\varepsilon > 0$,there exists $n_0 = n(\varepsilon) \in \mathbb{N}$,such that for any $n, m > n_0$ and for all $z \in K$,

$$|f_n(z) - f_m(z)| < \varepsilon$$

Suppose (i) does not hold, then there is some $\varepsilon' > 0$ such that for each $n \in \mathbb{N}$,there are $n_k > m_k > k$ with $\lim_{k \to \infty} m_k = \infty$ and some $z_k \in K$ such that

$$|f_{n_k}(z_k) - f_{m_k}(z_k)| > \varepsilon'$$

Since K is compact, $\{z_k\}$ has a convergent subsequence $\{z_{k_l}\}$ such that

$$\lim_{k \to \infty} z_{k_l} = z_0 \in K$$

For $\varepsilon/3$, the equicontinuity of $\mathfrak{F}$ ensures that there is some $\delta > 0$ such that whenever $|z_k - z_0| < \delta$

$$|f_n(z_k) - f_n(z_0)| < \varepsilon/3$$

for all n. Thus

$$\lim_{k\to\infty} |f_{n_k}(z_0) - f_{m_k}(z_0)| = 0$$

and for all $k > k_0$

$$|f_{n_k}(z_0) - f_{m_k}(z_0)| < \varepsilon/3$$

Moreover, for this δ,it's true $|z_k - z_0| < \delta$ for all $k > k_0$.Hence

$$\begin{aligned}\varepsilon' &< |f_{n_k}(z_k) - f_{m_k}(z_k)| \leq |f_{n_k}(z_k) - f_{m_k}(z_0)| + \\ &\quad + |f_{n_k}(z_0) - f_{m_k}(z_0)| + |f_{n_k}(z_k) - f_{n_k}(z_0)| \\ &< \varepsilon'\end{aligned}$$

which is a contradiction. ■

Proof. (Direct proof) Target: To show

$$f_n \rightrightarrows g$$

Let K be any compact subset of U,then there is some $r > 0$ such that for any $z \in K$,

$$B(z, r) \subseteq U$$

Obviously,$\mathcal{O} = \{B(z, r) : z \in K, r > 0\}$ forms an open cover of K and the compactness of K implies there exist some $m \in \mathbb{N}$ such that

$$K \subseteq \cup_{i=1}^{m} \{B(z_i, r) : B(z_i, r) \in \mathcal{O}\}$$

By the equicontinuity of $\mathfrak{F}$, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|z - z'| < \delta$

$$\left|f_n(z) - f_m(z')\right| < \varepsilon$$

for all $m, n \in \mathbb{N}$.(Correction by Dr. Bleecker: it should be $|f_n(z) - f_n(z')| < \varepsilon$ for all $n \in \mathbb{N}$) Let $\rho = \min\{r, \delta\}$. Then it's clear that $B(z, \rho) \subseteq U$ for all $z \in K$ and $\mathcal{O}_1 = \{B(z, \rho) : z \in K\}$ also forms an open cover of K and hence $K \subseteq \cup_{i=1}^{m_1} \{B(z_i, \rho) : B(z_i, \rho) \in \mathcal{O}_1\}$ for some $m_1 \in \mathbb{N}$. Since $(\not{f}_n)$ converges pointwise to g,there exist $n_0 \in \mathbb{N}$ such that for all $z_i, i = 1, ..m$,

$$|f_n(z_i) - g(z_i)| < \varepsilon$$

whenever $n > n_0$. Finally, for any $z \in K$, it's obvious that $z \in B(z_{i_0}, \rho)$ for some i_0 with $1 \leq i_0 \leq m_1$ and whenever $n > n_0$,

$$\begin{aligned}|f_n(z) - g(z)| &= |f_n(z) - f_n(z_{i_0}) - f_n(z) + f_n(z_{i_0}) + f_n(z) - g(z)| \\ &\leq |f_n(z) - f_n(z_{i_0})| + |f_n(z_{i_0}) - f_m(z)| + |f_n(z) - g(z)| < 3\varepsilon\end{aligned}$$

which means (f_n) converges uniformly to g on K. ■

Lemma 2.5 *A normal family $\mathfrak{F}$ of $C(U)$ is locally bounded in U.*

Proof. Let K be any compact subset of U. Then by (2.3), there exists $z_i \in K, i = 1, ...m$ and $r > 0$ such that

$$K \subseteq \cup_{i=1}^{m} \{\Delta(z_i, r) : z_i \in K, r > 0\} \subseteq U$$

Since every $f \in C(U)$ and K is compact, then $E_f = f(K) \subseteq \mathbb{C}$ is compact, $|f|$ is continuous on U and $G_f = |f|(K) \subseteq \mathbb{R}$. The inequality

$$||a| - |b|| < |a - b|$$

for all $a, b \in \mathbb{C}$ implies that $\{|f| : f \in \mathfrak{F}\}$ is also a normal family on U.Let

$$\Lambda = \left\{ \alpha_f = \max_{z \in K} \{|f(z)|\} : f \in \mathfrak{F} \right\}$$

Suppose $\bar{\Lambda}$ is not compact. Then theres exist $\mathcal{O} = \{B(z, r) : z \in \mathbb{C}, r > 0\}$ but for any $m \in \mathbb{N}$,

$$K \nsubseteq \cup_{i=1}^{m} \{B(z_i, r) : B(z_i, r) \in \mathcal{O}\}$$

Specifically, ince Since $\mathfrak{F}$ is normal, ■

Theorem 2.1 (Arzela-Ascoli) *A subfamily $\mathfrak{F} \subseteq \mathcal{C}(U)$ is normal iff it is both equicontinuous and pointwise bounded.*

Theorem 2.2 (Montel) *A subfamily $\mathfrak{F} \subseteq C(U)$ is normal iff it is locally bounded on U.*

Proof. Use Arezla-Ascoli theorem and Cauchy integral formula. ■

Lemma 2.6 *If G is open in $\mathbb{C}$ then there is a sequence (K_n) of compact subsets of G such that*

1. $G = \cup_{n=1}^{\infty} K_n$
2. $K \subseteq G$ and K compact implies $K \subseteq K_n$ for some n
3. Every component of $T_n = \widehat{\mathbb{C}} \sim K_n$ contains a component of $T = \widehat{\mathbb{C}} \sim G$ (Here $\mathbb{C}_\infty = \widehat{\mathbb{C}}$ is the Riemman Sphere), that is,

   let $\mathfrak{C}_1, \mathfrak{C}_2$ be the set of components of T_n and T,then $\mathfrak{C}_2 \preccurlyeq \mathfrak{C}_1$,i.e.,$\mathfrak{C}_2$ is finer than $\mathfrak{C}_1$

Proof. Apply the basic trick by for each $n \in \mathbb{N}$ define

$$K_n = \{z \in \mathbb{C} : |z| \leq n\} \cap \left\{z \in \mathbb{C} : d(z, \mathbb{C} \sim G) \geq \frac{1}{n}\right\}$$

Then K_n is compact. For

$$H_n = \{z \in \mathbb{C} : |z| < n+1\} \cap \left\{z \in \mathbb{C} : d(z, \mathbb{C} \sim G) > \frac{1}{n+1}\right\}$$

it's clear that H_n is open and

$$K_n \subseteq H_n \subseteq K_{n+1}$$

Then

$$G = \cup_{n=1}^{\infty} K_n = \cup_{n=1}^{\infty} \operatorname{Int} K_n$$

since

$$z \in \mathbb{C} : d(z, \mathbb{C} \sim G) \geq \frac{1}{n} \Leftrightarrow \exists \rho > 0 \text{ s.t. } B(z, \rho) \subseteq G$$

Further, for any compact K with $K \subseteq G$,it's clear that $K \subseteq \cup_{n \in \mathbb{N}} \operatorname{Int} K_n$ and $K \subseteq K_{n_0}$ for some $n_0 \in \mathbb{N}$

To see (3), Let E be the unbounded component of $\widehat{\mathbb{C}} \sim K_n \subseteq \widehat{\mathbb{C}} \sim G$ and let F be the unbouned compoment of $\widehat{\mathbb{C}} \sim G$.Then $E \supseteq \{z \in \mathbb{C} : |z| > n\}$, $\infty \in E$ and $E \supseteq F$ since $\widehat{\mathbb{C}} \sim K_n \subseteq \widehat{\mathbb{C}} \sim G$.So if D is a bounded component of $\widehat{\mathbb{C}} \sim K_n$ it contains some z with $d(z, \mathbb{C} \sim G) < \frac{1}{n}$. But, then there is some $w \in \mathbb{C} \sim G$ with $|z - w| < \frac{1}{n}$ and $z \in B(w; \frac{1}{n}) \subseteq \widehat{\mathbb{C}} \sim K_n$.Since disks are connected and $z \in D$ with $D \subseteq \widehat{\mathbb{C}} \sim K_n$ being a component,then $B(w; \frac{1}{n}) \subseteq D$.If D_1 is the component of $\widehat{\mathbb{C}} \sim G$ that contains w it follows that $D_1 \subseteq D$ ■

Remark 3 *This works for any complete metric space.*

4 Metrization of $\mathcal{H}(U)$

Let G and K_n be as given (2.6) and $f, g \in C(G, \Omega)$.Define

$$\rho_n(f, g) = \sup \{d(f(z0, g(z)) : z \in K_n\}$$

and

$$\rho(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)}$$

Theorem 4.1 $(C(G, \Omega), \rho)$ *is a complete, locally convex metric space*

Remark 5 *Consult Kosako Yoshida for Locally convex metrix space*