Real Analysis 1992

Problem 1 *(a) State the Weierstrass Approximation Theorem for continuous functions on* $[0,1]$ *;(b) Prove that if* $f \in L([0,1])$ *and*

$$\int_{[0,1]} x^n f(x)\,dx = \frac{1}{n+1}$$

for any $n \in \{0,1,...\}$ *, then*

$$f = 1 \ a.e.$$

Proof. (a) Any continuous function defined on $[0,1]$ can be uniformly approximately by polynomials defined on $([0,1])$

(b) Define

$$F(x) = \int_{[0,x]} (f(t) - 1)\,dt$$

for $x \in [0,1]$, then F is absolutely continuous on $[0,1]$ since $f(t) - 1 \in L([0,1])$ and

$$F' = f - 1 \text{ a.e.}$$

Further, $g_n(x) := x^n$ being absolutely continuous for all $n \in \mathbb{N}$ allows integration by parts, that is,

$$\begin{aligned} 0 &= \int_{[0,1]} x^n (f(x) - 1)\,dx = (g_n(x) F(x))|_0^1 - \int_{[0,1]} n x^{n-1} F(x)\,dx \\ &= F(1) - \int_{[0,1]} n x^{n-1} F(x)\,dx \end{aligned}$$

but

$$F(1) = \int_{[0,1]} (f(t) - 1)\,dt = 0$$

thus

$$0 = n \int_{[0,1]} x^{n-1} F(x)\,dx$$

for all $n \in \mathbb{N}^*$ and for any polynomial $p(x) := \sum_{k=0}^m a_k x^k, m \in \mathbb{N}, a_k \in \mathbb{R}$

$$0 = \int_{[0,1]} p(x) F(x)\,dx = \sum_{k=0}^m a_k \int_{[0,1]} x^k F(x)\,dx \tag{1}$$

By the Weierstarss Approximation theorem, there exists a sequence of polynomials $\{p_k(x)\}_{k \in \mathbb{N}^*}$ defined on $[0,1]$ such that

$$\lim_{k\to\infty} \sup_{x\in[0,1]} |F(x) - p_k(x)| = 0 \tag{2}$$

Consequenlty by (1),(2),

$$0 \equiv \lim_{k\to\infty} \int_{[0,1]} p_k(x) F(x)\,dx = \int_{[0,1]} F(x) F(x)\,dx = \int_{[0,1]} F^2(x)\,dx$$

which implies

$$F \equiv 0$$

since F is continuous, whence

$$0 \equiv F' = f - 1 \text{ a.e}$$

i.e., $f = 1$ a.e.. ■

Problem 2 *Can you please suggest to me a way of solving this via functional analysis, since $P([0,1])$ (polynomials) is dense in $C([0,1])$ and the latter is dense in $L^p([0,1]), 1 \leq p < \infty$ in L^p normal and for each $f \in L([0,1])$*

$$\Phi_f(g) = \int_{[0,1]} f(x)\, g(x)\, dx$$

induces a functional (at least) on $L([0,1])$ and Φ_f is completed decided on a dense subset of $L([0,1])$

Please check this proof for me and suggest a functional way of doing it. Thanks.

I feel deeply sorry for making no progress in differential geometry.