Math 242 (April 18,2008)

1 Summary

Binomial expansion and the period of a pendulum will be discussed today. Please turn in your answers at the end of the class. Thanks

2 Binomial Expansion

For a non-negative integer n and any real numbers a, b, we've

$$(a+b)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^k b^{n-k} \tag{1}$$

and that's the binomial theorem leant in high school.

Actually, (1) can be extended to be defined to any real number α with similar repersentation on the right side but with some restrictions in order to make sense. Generally, we have

$$(1+x)^a = \sum_{k=0}^{\infty} \left(\begin{array}{c} \alpha \\ k \end{array} \right) x^k$$

for all x with $|x| < 1$, where

$$\left(\begin{array}{c} \alpha \\ k \end{array} \right) = \begin{cases} \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & if \quad k \geq 1 \\ 1 & if \quad k = 0 \end{cases}$$

One extremely important thing is that, when a power series expansion is derived from a given function, the function this power series defines may have smaller domain thant the original function. Hereunder is an easy but interesting example. Let

$$f(x) = \frac{1}{1+x}$$

(your turn please:

Q1: What's the domain of $f(x)$?

Q2: What's the Maclaurin series of $f(x)$?

Q3: What's the radius of convergence R of the power series expansion you give? Is $R = 1$?

3 Pendulum

The period of a pendulum with length L that makes a mximus angle θ_0 with the vertical is

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} \tag{2}$$

(the above equation contains an elliptical integral) where $k = \sin(\theta_0/2)$ and g is the acceleration due to gravity.

1. Expand the integrand as a binomial series and use the result that

   $$\int_0^{\pi/2} \sin^{2n} x dx = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2}$$

   to show that

   $$T = 2\pi\sqrt{\frac{L}{g}} \left[1 + \frac{1^2}{2^2}k^2 + \frac{1^2 3^2}{2^2 4^2}k^4 + \frac{1^2 3^2 5^2}{2^2 4^2 6^2}k^6 + \cdots\right] \tag{3}$$

   (In order to get (3), you may assume that the order of integration and summation can be interchanged. **Q4**: make a guess on a condition under which such an interchange is admissible). If θ_0 is sufficiently small

   $$T \approx 2\pi\sqrt{\frac{L}{g}} \tag{4}$$

   (**Q5**: Explain why (4) is correct when θ_0 is sufficiently small; **Q:** Can you give a better approximate of T based on (3) when θ_0 is sufficiently small?)

2. Find a bound for all the coefficients of the summands with positive powers in the bracket shown in (3) and majorize the series in the bracket by some geometric series and show that

   $$2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1^2}{2^2}k^2\right) \leq T \leq 2\pi\sqrt{\frac{L}{g}} \frac{4 - 3k^2}{4 - 4k^2} \tag{5}$$

3. Use (5) to estimate the period of a pendulum with $L = 1$ meter and $\theta_0 = 10°$. How does it compare with the estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$. What if $\theta_0 = 42°$?

Finally, have a nice weekend :)