Math 242 (March 7, 2008)

Approximate Integration

Please turn in your answers at the end of today's session. Thanks.

1 Reminders

1. Please turn in your answers to the assignments in time and if you are not able to come to class, please email me in advance at *chee@math.hawaii.edu* and I'll email you the contents to be covered

2. Please be familar with basic commands in MatLab and Derive and be adapted to the way in which a Math software thinks and solves problems

2 Introduction

In cases where the integrand does NOT have an explicit or elementary representation of its primitive, numerical methods should be use to approximate the exact integral. Hereunder are a couple of examples which defy usual analytic attempts:

1. $\int_0^\infty \frac{\sin x}{x} dx$

2. $\int_0^1 \frac{e^{x^2}}{x} dx$

3. $\int_{-1}^1 \sqrt{1+x^3} dx$

Although there are various numerical approximations to a given integral, they all stem from a simple fact that

By the definition of Riemann Integral, for any function $f(x)$ integrable on the finite interval $I = [a, b] \subseteq \mathbb{R}$,

$$\int_a^b f(x)\, dx = \lim_{\max\{\Delta x_i\} \to 0} \sum_{k=1}^{n} f(\zeta_k)\, \Delta x_k$$

holds for any partition

$$\mathcal{P} = \{x_0 = a, \cdots, x_n = b : x_k < x_{k+1}, 1 \le k \le (n-1)\}$$

that splits I into n subintervals $I_k = [x_k, x_{k+1}]$ and any $\zeta_k \in I_k$, where $\Delta x_k = x_{k+1} - x_k$ and $\lim_{n\to\infty} \max\{\Delta x_i\} = 0$.

Thus for n sufficiently large, it's clear that

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f(\zeta_k)\,\Delta x_k$$

for any such partition and any $\zeta_k \in I_k$.

Specifically, for $\mathcal{P}$ that splits I into equal-length subintervals (ie., $\Delta x_k = x_{k+1} - x_k = (b-a)/n$) with ζ_k specifically chosen, the following methods are induced

1. Midpoint Rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n} \sum_{k=1}^{n} f\left(\frac{x_k + x_{k+1}}{2}\right)$$

   and

2. Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n} \left[f(a) + f(b) + 2\sum_{k=1}^{n} f(x_k)\right]$$

3 Practice

Problem 1 *Use Trapezoidal and midpoint Rule to approximate*

$$\int_1^2 \frac{1}{x}dx$$

Solution. 1. Divide $I = [1,2]$ into 1000 equal-length subintervals, then $\Delta x_k = \dfrac{1}{1000}$ and $x_k = 1 + \dfrac{k}{1000}$

2. Summing SOME $f(x_k) = 1/\left(1 + \dfrac{k}{1000}\right)$ and then multiply the sum by Δx_k.

3. Matlab commands for Trapezoidal Rule :

> n=1.2:0.2:1.8;

>v=1+1/2;

>s=0;

```
> for n=1:0.2:1.8
>s=s+2/n
>end
t=0.1*(s+v)
```

Assignment: Please explain what these commands mean. Write your answers after each command.

Now, increase the number of equilength subintervals to 1000 and execute the following commands:

```
>format long
>n=1.0001:0.0001:1.999;
>v=1/1+1/2;
>s=0;
>for n=1.0001:0.0001:1.9999
>s=s+2/n
>end
>t=0.0005*(s+v)
```

Assignment: Please explain the meanings of the quantities in the commands. Write your answer after each command.

This will give a more accurate approximation, thus

$$\ln 2 = \int_1^2 \frac{1}{x} dx \approx 0.69314724305994$$

4. Matlab Commands for Midpoint Rule

```
> n=1.1:0.2:1.9;
>s=0;
>for n=1.1:0.2:1.9
>s=s+1/n
>end
>t=(1/5)*(s+v)
```

Again for the equilength partition of I into 1000 subintervals, the approximation is achieved by

```
>n=1.001:0.001:1.999;
>s=0;
>for n=1.001:0.001:1.999
>s=s+1/n
>end
>t=(1/1000)*s
```

Thus

$$\ln 2 = \int_1^2 \frac{1}{x} dx \approx 0.69239724305994$$

■

Problem 2 *Approximate*

$$\int_0^1 \frac{\sin x}{x} dx$$

by Trapezoidal rule.

Solution. Let $f(x) = \frac{\sin x}{x}$ and divide $[0, 1]$ into 1000 equilength subintervals $I_k = \left[\frac{k}{1000}, \frac{k+1}{1000}\right]$ for $0 \le k \le 1000$.Then

Q1: What is the value of the the leftend point and what is the value of the function at that point?

Q2: What is the step length of the array of values the divides $[0, 1]$ and what is the initial value of the sum in interation?

Q3: What is the key MatLab command for iteration?

A4: Write down the commands executed to solve this problem. ■

Please write down your answers in the space given hereunder.