Math 252 (April 23, 2008)

1 Summary

Certain explications on how to solve second order linear equation with constant coefficients will be presented as supplementary to text.

2 Derivation of the Solution

1. General Motivation

   As introduced in previous lab sessions, function

   $$f(t) = e^t$$

   has the property that

   $$f^{(n)}(t) \equiv e^t$$

   for all $n \in \mathbb{N}$ and is never zero for all $t \in \mathbb{R}$.Thus it's used to test whether a equation of the following form

   $$\sum\nolimits_{k=0}^{n} a_0 x^{(n)}(t) = 0 \tag{1}$$

   has a solution of the form

   $$x(t) = \sum\nolimits_{k=0}^{n} b_k e^{c_k t} \tag{2}$$

   or not, where in (1), (2) $a_k, b_k, c_k, k = 0, \cdots, n$ are some real constants and $y^{(0)}(t) = y(t)$ .This is covered in the textbook. Actually, the following theorem is derived

   Theorem 1 *For*

   $$a_0 \frac{d^2}{dt^2} x(t) + a_1 \frac{d}{dt} x(t) + a_2 = 0 \tag{3}$$

   where $a_k, k = 0, \cdots, 2$ are real constants. If it's characteristic equation

   $$a_0 \lambda^2 + a_1 \lambda + a_2 = 0$$

   has two roots λ_1, λ_2,then

   (a) when $a_1^2 - 4a_0 a_1 > 0$,(3) has real solution

   $$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

   (b) when $a_1^2 - 4a_0 a_1 = 0$,(3) has real solution

   $$x(t) = (c_1 + c_2) e^{\lambda_1 t}$$

(c) *when* $a_1^2 - 4a_0a_1 < 0$ *and* $\lambda_1 = \alpha + \beta i$,*(3) has real solution*

$$x(t) = e^{\alpha t}\left(c_1 \cos \beta t + c_2 \sin \beta t\right)$$

2. Nonhomogenous second-oder equation with constant coefficients

   The details of how to prove the above theorem is embedded in the derivation of solutions to

$$a_0 \frac{d^2}{dt^2} x(t) + a_1 \frac{d}{dt} x(t) + a_2 = f(t) \tag{4}$$

which is a variation (but may be a lot more difficult to solve) of (3) but has the same characteristic equation

$$a_0 \lambda^2 + a_1 \lambda + a_2 = 0 \tag{5}$$

Now suppose (5) has two different roots λ_1, λ_2, then $a_0 \neq 0$.Now setting

$$y = \frac{dx}{dt} - \lambda_1 x \tag{6}$$

gives

$$a_0 \left\{ \frac{dy}{dt} - \lambda_2 y \right\} = f(t)$$

that is

$$\frac{d}{dt}\left\{e^{-\lambda_2 t} y\right\} = \frac{1}{a_0} f(t) e^{-\lambda_2 t}$$

So

$$y = c_2' e^{\lambda_2 t} + \frac{1}{a_0} \int_0^t e^{\lambda_2 (t-s)} f(s)\, ds \tag{7}$$

where c_2' is an arbitrary constant. Similarly, setting

$$z = \frac{dx}{dt} - \lambda_2 x \tag{8}$$

gives

$$z = c_1' e^{\lambda_1 t} + \frac{1}{a_0} \int_0^t e^{\lambda_1 (t-s)} f(s)\, ds \tag{9}$$

Now subtract (7) from (9) and divide the resultant equation by $(\lambda_1 - \lambda_2)$,then

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \frac{1}{a_0} \int_0^t \frac{e^{\lambda_1 (t-s)} - e^{\lambda_2 (t-s)}}{\lambda_1 - \lambda_2} f(s)\, ds \tag{10}$$

where $c_1 = c_1' / (\lambda_1 - \lambda_2), c_2 = c_2' / (\lambda_1 - \lambda_2)$ are arbitrary complex constants. Usually, it's set that

$$h(t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}$$

and then (10) becomes

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \frac{1}{a_0} \int_0^t h(t-s) f(s) \, ds \tag{11}$$

This method is the so called **variation of parameters**

From (11) it's clear that such a solution is the superposition of the general solution

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

of equation

$$a_0 \frac{d^2}{dt^2} x(t) + a_1 \frac{d}{dt} x(t) + a_2 = 0$$

and a particular solution

$$x = \frac{1}{a_0} \int_0^t h(t-s) f(s) \, ds$$

of

$$a_0 \frac{d^2}{dt^2} x(t) + a_1 \frac{d}{dt} x(t) + a_2 = f(t)$$

3 Practice

Problem 2 *Find the real solutions to*

$$\frac{d^2}{dt^2} \theta(t) + \frac{g}{l} \theta(t) = 0$$

where g, l are given positive constants.

Problem 3 *Find the solutions to*

$$\frac{d^2}{dt^2} x(t) + \omega_0^2 x = A \sin \omega t$$

where ω_0, A are given constants.