Math 252 Lab (Feb 06, 2008)

1 Contents Covered

1. L'Hospital Rule
2. Infinitesimals
3. Hyperbolic functions
4. Taylor Expansion (Supplementary)

2 Infinitesimals

Let's start with a simple example, say, x. It should be noted that x here is regarded as variable (ranging over all admissible vaules). We know that

$$\lim_{x\to 0} x = 0$$

Also, we know that

$$\lim_{x\to 0} \sin x = 0$$

We call $x, \sin x$ *infinitesimals*. Intuitively, any quantity that tends to zero is called an infinitesimal.

One natural question one tends to ask is that,

> Amongst x, $\sin x$, which goes to zero faster as x tends to 0?

This is equivalent to solve

$$\lim_{x\to 0} \frac{\sin x}{x} = ?$$

which may take a while for the exact answer to appear.

3 Practice

1. Graph

   $$f(x) = \frac{\sin x}{x}$$

   and the find out

   $$\lim_{x\to 0} \frac{\sin x}{x}$$

2. Given the following pair of functions

$$f(x) = \tan x \quad \text{and} \quad g(x) = \sin x$$

graph

$$\frac{f(x)}{g(x)}$$

and

$$\frac{f'(x)}{g'(x)}$$

What can you say about the relation between $\lim_{x\to 0} \frac{f(x)}{g(x)}$ and $\frac{f'(0)}{g'(0)}$

4 Quiz Problems

1. Graph the function $y = \tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x - e^{-x}}$ and find out

$$\lim_{x\to\infty} \tanh x$$

and

$$\lim_{x\to 0} \tanh x$$

(Hint: graph the auxiliary functions $y = 1, y = -1$)

2. Graph the function $y = x^x$ and find out

$$\lim_{x\to 0+} x^x$$

(Hints: $y = x^x = \left(e^{\ln x}\right)^x$. Graph the auxiliary function $f(x) = x \ln x$ and find out $\lim_{x\to 0+} x \ln x$)

3. Show that

$$f(x) = \begin{cases} |x|^x & if \quad x \neq 0 \\ 1 & if \quad x = 0 \end{cases}$$

is continuous at $x = 0$ with the help of graph and l'Hospital Rule. (Extra credit: Is f differentiable at $x = 0$?)

5 Taylor Expansion

If a function $f(x)$ is differentiable up to the nth order at some neighborhood of a given point x_0 in the domain of f, then

$$f(x) = \sum\nolimits_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^n + \varepsilon(x, x_0)$$

where $f^{(0)}(x) = f(x)$ and $\lim_{x \to x_0} \varepsilon(x, x_0) = 0$.

Once Taylor Expansion is known to you, you'll have a handy tool to cope with *indeterminate forms* such as $\infty^0 \sim 0^\infty$, $\dfrac{0}{0} \sim \dfrac{\infty}{\infty}$.