Power Series

Since HW assignments are feedbacks of how you learnt the contents and of part of the total credits, please turn in due HW assignment in time.

Please pay attention and try your best to understand the meanings of the sentences in bold face. :)

1 Preliminaries

It'll be assumed that the attendants have acquired the basics of how to find limits of sequences and of the ratio test, which says:

Given any series $\sum_{n=0}^{\infty} a_n$. If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r \tag{1}$$

then

$$\begin{cases} r<1 \text{ implies the convergence of} \sum_{n=0}^{\infty} a_n \\ r=1 \text{ holds the convergence of } \sum_{n=0}^{\infty} a_n \text{ pending} \\ r>1 \text{ implies the divergence of } \sum_{n=0}^{\infty} a_n \end{cases}$$

As was pointed out

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$$

poses a situation where other tests or delicate computations are needed.

Remark 1 *Sometimes* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ *does NOT exist. Here is an example. Define*

$$a_n = \begin{cases} 2^{-n} & if \quad n \text{ is even} \\ (2/3)^n & if \quad n \text{ is odd} \end{cases}$$

Then for the series $\sum_{n=0}^{\infty} a_n$ *the ratio*

$$\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} \dfrac{(2/3)^{2k+1}}{(1/2)^{2k}} = \dfrac{2}{3}\left(\dfrac{4}{3}\right)^{2k} & if \quad n=2k \quad for \quad k\in\mathbb{N} \\ \dfrac{(1/2)^{2k+2}}{(2/3)^{2k+1}} = \dfrac{3}{2}\left(\dfrac{3}{4}\right)^{2k+1} & if \quad n=2k+1 \quad for \quad k\in\mathbb{N} \end{cases}$$

thus

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} \infty & if \quad n \text{ is even} \\ 0 & if \quad n \text{ is odd} \end{cases}$$

So, usually the ratio test takes this form

$$\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} r<1 \text{ implies the convergence of} \sum_{n=0}^{\infty} a_n \\ r=1 \text{ holds the convergence of } \sum_{n=0}^{\infty} a_n \text{ pending} \\ r>1 \text{ implies the divergence of } \sum_{n=0}^{\infty} a_n \end{cases}$$

2 Geometric Series

For

$$\sum\nolimits_{n=0}^{\infty} x^n \tag{2}$$

it's routine to set

$$s_m = \sum\nolimits_{n=0}^{m} x^n$$

Then

$$\begin{aligned} xs_m &= x\sum\nolimits_{n=0}^{m} x^n = \sum\nolimits_{n=1}^{m+1} x^n = x^{m+1} - 1 + \sum\nolimits_{n=0}^{m} \\ &= x^{m+1} - 1 + s_m \end{aligned}$$

and

$$s_m = \frac{1 - x^{m+1}}{1 - x} \tag{3}$$

Since

$$\lim_{m\to\infty} |x|^{m+1} = \begin{cases} 0 & if \quad |x| < 1 \\ 1 & if \quad |x| = 1 \\ \infty & if \quad |x| > 1 \end{cases}$$

then taking limits on both sides of (3), it's obtained

$$\sum\nolimits_{n=0}^{\infty} x^n = \lim_{m\to\infty} s_m = \lim_{m\to\infty} \frac{1 - x^{m+1}}{1 - x} = \begin{cases} \dfrac{1}{1-x} & if \quad |x| < 1 \\ \infty & if \quad |x| = 1 \\ \infty & if \quad |x| > 1 \end{cases}$$

The key idea is that for a geometric series with $|x| < 1$, it's always true that

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{x^n}\right| = |x| < 1$$

thus for such x, the series converges.

3 General Power Series

As long as the properties of the geometric series is understood, it's admissible to move on to general power series since **all test criteria for power series stem from comparing the tail series with (the tail of) the geometric series**.

Series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{4}$$

is called a power series centered at x_0.

For any fixed x, (4) is just a series and the ratio test could be attempted to test its convergence. In case (4) is convergent to, say, y, for a fixed x,then all such x and all corresponding y establish a functional relation between these x and the limits. **Thus, a function $f(x)$ could be defined as**

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

for all x such that (4) converges.

Theorem 2 *For any power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$, there's some R with $0 \leq R \leq +\infty$ such that for any $0 \leq R' < R$ and x such that $|x - x_0| \leq R'$, $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ absolutely and uniformly; while for all x with $|x - x_0| > R$, $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ diverges; and for $|x - x_0| = R$, $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ may converge or diverge, where R is defined by*

Cauchy-Hadamard Formula

$$R = \frac{1}{\lim_{n\to\infty} \sup |a_n|^{1/n}}$$

Theorem 3 *If*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \text{ exists}$$

then

$$R = \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

Remark 4 *Note that taking $a_n \equiv 1$ and $x_0 = 0$, then (4) reduces to (2).*

4 Practice

Example 5 *(Page 615) Find the radius of convergence and interval of convergence of the series*

$$\sum\nolimits_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} \tag{5}$$

Solution. Method 1: Let $b_n(x) = \dfrac{(-3)^n x^n}{\sqrt{n+1}}$ (which means, x is taken as fixed temporarily). Then

$$\begin{aligned}
\left|\frac{b_{n+1}(x)}{b_n(x)}\right| &= \left|\frac{\dfrac{(-3)^{n+1}x^{n+1}}{\sqrt{(n+1)+1}}}{\dfrac{(-3)^n x^n}{\sqrt{n+1}}}\right| \\
&= \left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{(n+1)+1}}\frac{\sqrt{n+1}}{(-3)^n x^n}\right| \\
&= 3\sqrt{\frac{1+(1/n)}{1+(2/n)}}\,|x| \to 3|x|
\end{aligned}$$

as $n \to \infty$. By the ratio test, (5) converges if $3|x| < 1$ and diverges if $3|x| > 1$. Thus it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$ and the radius of convergence $R = \frac{1}{3}$.

Method 2: Rewrite $\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}}$ as $\sum_{n=0}^{\infty}\frac{(-3)^n}{\sqrt{n+1}}x^n$ and let $a_n = \frac{(-3)^n}{\sqrt{n+1}}$, then (5) is identically

$$\sum_{n=0}^{\infty} a_n x^n$$

By (3), the radius of convergence

$$\begin{aligned}
R &= \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \frac{1}{\lim_{n\to\infty}\left|\frac{(-3)^{n+1}}{\sqrt{(n+1)+1}}\frac{\sqrt{n+1}}{(-3)^n}\right|} \\
&= \frac{1}{3\lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{\sqrt{(n+1)+1}}\right|} = \frac{1}{3}\lim_{n\to\infty}\left|\sqrt{\frac{1}{n+1}+1}\right| = \frac{1}{3}
\end{aligned}$$

Method 3: By Cauchy-Hadamad formula

$$\begin{aligned}
R &= \frac{1}{\lim_{n\to\infty}\sup|a_n|^{1/n}} = \frac{1}{\lim_{n\to\infty}\sup\left|\frac{(-3)^n}{\sqrt{n+1}}\right|^{1/n}} \\
&= \frac{\lim_{n\to\infty}\left(\sqrt{n+1}\right)^{1/n}}{\lim_{n\to\infty}|-3|} = \frac{1}{3}
\end{aligned}$$

where the fact that

$$\lim_{n\to\infty}\left(\sqrt{n+1}\right)^{1/n} = 1$$

is used.

Thus the interval of convergence **contains** $\left(-\frac{1}{3}, \frac{1}{3}\right)$ (since the endpoints are not checked yet). If $x = -\frac{1}{3}$, then

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-3)^n \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} &= \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} \geq \int_0^{\infty} \frac{1}{\sqrt{x+1}} dx \\
&= \lim_{r \to +\infty} \int_0^r \frac{1}{\sqrt{x+1}} dx = \lim_{r \to +\infty} \left((x+1)^{1/2} \Big|_0^r \right) \\
&= \lim_{r \to +\infty} \left((r+1)^{1/2} - 1 \right) = +\infty
\end{aligned}$$

and it diverges. But, if $x = \frac{1}{3}$, then

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

converges since

1. $\frac{1}{\sqrt{n+1}} > 0$
2. $\left\{ \frac{1}{\sqrt{n+1}} \right\} \downarrow$
3. $\lim_{n \to \infty} \frac{1}{\sqrt{n+1}} = 0$

Consequently, the interval of convergence is

$$I = \left(-\frac{1}{3}, \frac{1}{3} \right]$$

■

5 Homework Assignments

1. (Prob 19 on Page 617) If k is a positive integer, find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$$

(Hint: It's easier to use ratio test)

6 Ratio Test Revisited

Given any series $\sum_{n=0}^{\infty} a_n$ and if

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r \tag{6}$$

then

$$\begin{cases} r < 1 \text{ implies the convergence of} \sum_{n=0}^{\infty} a_n \\ r = 1 \text{ holds the convergence of } \sum_{n=0}^{\infty} a_n \text{ pending} \\ r > 1 \text{ implies the divergence of } \sum_{n=0}^{\infty} a_n \end{cases}$$

Brief Proof: Let's first consider the third case. Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r > 1$. Let $\delta = r - 1$ and pick ρ with $0 < \rho < \delta$. Then by the definition of limit, for this ρ, there exist some n_0 such that

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - r\right| < \rho$$

for all $n > n_0$, which means

$$\left|\frac{a_{n+1}}{a_n}\right| > r - \rho > 1 \tag{7}$$

and $|a_{n+1}| > (r-\rho)\,|a_{n_0}|$ for all $n > n_0$. Thus for any $m = n + p$ with $n > n_0$ and $p \in \mathbb{N}$

$$|a_{n+p}| > (r-\rho)^p\,|a_{n_0}| \tag{8}$$

Take limits on both sides of (8), then

$$\lim_{p\to\infty} |a_{n+p}| > \lim_{p\to\infty} (r-\rho)^p\,|a_{n_0}| = +\infty$$

which implies $\sum_{n=0}^{\infty} a_n$ can NOT be convergent. (Why?)

For the first case. Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r < 1$. Let $\delta = 1 - r$ and pick ρ with $0 < \rho < \delta$. Then by the definition of limit, for this ρ, there exist some n_0 such that

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - r\right| < \rho$$

for all $n > n_0$, which means

$$\left|\frac{a_{n+1}}{a_n}\right| < r + \rho < 1 \tag{9}$$

and $|a_{n+1}| < (r+\rho)\,|a_{n_0}|$ for all $n > n_0$. Thus for any $m = n + p$ with $n > n_0$ and $p \in \mathbb{N}$

$$|a_{n+p}| < (r+\rho)^p\,|a_{n_0}| \tag{10}$$

and

$$\sum\nolimits_{p=1}^{\infty} |a_{n+p}| < \sum\nolimits_{p=1}^{\infty} (r+\rho)^p \, |a_{n_0}| = |a_{n_0}| \sum\nolimits_{p=1}^{\infty} (r+\rho)^p \tag{11}$$

But $(r+\rho) < 1$ implies that

$$\sum\nolimits_{p=1}^{\infty} (r+\rho)^p = \frac{1}{1-(r+\rho)} - 1 = \frac{r+\rho}{1-(r+\rho)}$$

Thus

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{n_0} a_n + \sum\nolimits_{p=1}^{\infty} a_{n+p}$$

is such that

$$\begin{aligned} \sum_{n=0}^{\infty} |a_n| \quad &\le \quad \sum_{n=0}^{n_0} |a_n| + \sum\nolimits_{p=1}^{\infty} |a_{n+p}| \qquad (12) \\ &\le \quad \sum_{n=0}^{n_0} |a_n| + |a_{n_0}| \sum\nolimits_{p=1}^{\infty} (r+\rho)^p \le \sum_{n=0}^{n_0} |a_n| + \frac{r+\rho}{1-(r+\rho)} \end{aligned}$$

Note in (12) n_0 is finite, thus $\sum_{n=0}^{n_0} |a_n|$ is finite (why?). Consequently $\sum_{n=0}^{\infty} |a_n|$ converges **ABSOLUTELY** and it must converge.

For the second case, $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r = 1$ means there may be infinitely many terms in the sum that are as close as desired to one or so are their absolute values, which means the tail of the sequence may not be majored by a geometricl series.