Complex Analysis 2003

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Abstract

When I was a child, I wanted to be a pilot. Later on in junior high school, I wanted to work for NASA, since I could be part of the space program and I'll be happy with it.

Never would I be so inspired if I did not come to US and particularly, if I did not meet my favorite professor here at UHM.

1 Notations

$\mathbb{C}$ the complex plane; $D = \{z \in \mathbb{C} : |z| < 1\}$; $\mathbb{N} = \{1, 2, 3, ...\}$; $\mathfrak{R}z$ the real part of z

2 Problems and Solutions

Problem 2.1 *Does there exist a function f having both the following properties: (i) f is holomorphic on D; (ii) $\lim_{n\to\infty} |f(z_n)| = \infty$ whenever $(z_n)_{\mathbb{N}}$ is a sequence in D and $\lim_{n\to\infty} |z_n| = 1$?*

Solution. Let $E := \{z \in D : f(z) = 0\}$. Suppose that there is an infinite sequence $\{z_n\}_{n\in\mathbb{N}}$ in E. Since the closed disk $\overline{D}$ is compact, $\{z_n\}_{n\in\mathbb{N}}$ must have a convergent subsequence $\{z_{n_k}\}_{n_k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} z_{n_k} = z_0 \in \overline{D}.$$

If $z_0 \in D$, then z_0 is a zero of f by the continuity of f. Since z_0 is not an isolated zero and D is connected, $f = 0$ on D. That contradicts (ii). If $|z_0| = 1$, then $\lim_{z_{n_k}\to z_0} |f(z_{n_k})| = \lim_{z_n\to z_0} 0 = 0$ which also contradicts (ii). Thus, E is finite, say

$$E = \{z_i \in D : i = 1, .., n\}.$$

Let the multiplicity of $z_i \in E$ be $p_i \in \mathbb{N}$. Then f can be written as

$$f(z) = g(z)\Pi_{i=1}^{n}(z - z_i)^{p_i}, \tag{1}$$

where g is holomorphic on D and never zero. Then we can define

$$h(z) = \frac{1}{g(z)} \in \mathcal{H}(D).$$

If $z' \in \partial D$, then there is a sequence $(z_k)_{\mathbb{N}}$ in $D \setminus E$ with $z' = \lim_{k\to\infty} z_k$. We then have

$$\lim_{k\to\infty} |h(z_k)| = \lim_{k\to\infty} \frac{1}{|g(z_k)|} = \lim_{k\to\infty} \frac{\Pi_{i=1}^{n} |z_k - z_i|^{p_i}}{|f(z_k)|} = 0,$$

since $\lim_{k\to\infty} |f(z_k)| = \infty$. Thus, h extends continuously to $\overline{D}$ and its extension is 0 on ∂D. By the maximum modulus principle, $h = 0$ on $\overline{D}$, which contradicts $h(z) = \frac{1}{g(z)}$. Thus, f does not exist. ■