Math 140  Lecture 3

Remainder: Gateway Exam next week.

Factoring and roots

THEOREM. If \( a > 0 \), \( x^2 - a = (x - \sqrt{a})(x + \sqrt{a}) \)
But \( x^2 + a \) has no roots and can’t be factored any more.

DIVISION LAW. If \( p(x)/d(x) \) has quotient \( q(x) \) and remainder \( r(x) \) then
\[
\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.
\]
Multiply by \( d(x) \) to get \( p(x) = d(x)q(x) + r(x) \).

\( d(x) \) divides into \( p(x) \) evenly iff the remainder is 0 iff
\( p(x) = d(x)q(x) \) iff \( d(x) \) is a factor of \( p(x) \).

If \( d(x) \) is a factor of \( p(x) \), the other factor of \( p(x) \) is \( q(x) \), the quotient of \( p(x)/d(x) \).

- Given \( p(x)/d(x) \), divide to get the quotient \( q(x) \) and remainder \( r(x) \). Write the answer in division law form:
\[
p(x) = d(x)q(x) + r(x).
\]

- \( x^2 + 1 \), \( x^2 + 1 = x^2 + x + 1 + \frac{2}{x^2} \), \( x^3 + 1 = (x - 1)(x^2 + x + 1) + 2 \)

Check that the answer is correct for \( x = 0 \).
For \( x = 0 \), we get \( 0 + 1 = (-1)(0 + 0 + 1) + 2 \), \( 1 = 1 \). \( \checkmark \)

- \( x - \) intercept. \( a \) is a root or zero of \( p(x) \) iff \( p(a) = 0 \).

THEOREM. \( a \) is a root of \( p(x) \) iff \( (x - a) \) is a factor of \( p(x) \). Note, rewrite \( (x + 3) \) as \( -(x - 3) \).

To find all roots of \( p(x) \), completely factor \( p(x) \).

Factor the polynomial and find all roots.

- \( x + 2 \)
  - Root: \(-2\)
- \( x^2 + 2 \)
  - Fully factored as is, no roots.
- \( x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2}) \) Rts: \(-\sqrt{2} \), \( \sqrt{2} \)
- \( x^2 - 4x + 4 = (x - 2)^2 \)
  - One repeated factor. Root: \(2\)
- \( x^3 + 5x^2 + 8x + 4 \) given that \(-1 \) is a root.
  \[
  (x - a) = (x - (-1)) = (x + 1) \) . we divide by \((x + 1)\).
  \[
  x^3 + 5x^2 + 8x + 4 \quad (x + 1) = x^2 + 4x + 4 = (x + 2)(x + 2) = (x + 2)^2
  \]
  \[
  x^3 + 5x^2 + 8x + 4 \quad (x + 1) = x^2 + 4x + 4 = (x + 2)(x + 2) = (x + 2)^2
  \]
  Roots: \(-2\), \(-1\).
- \( x^3 - x^2 - 2x + 2 \) given that \(1 \) is a root.
  \[
  x^3 - x^2 - 2x + 2 \quad (x - 1) = x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2}).
  \]
  Roots: \(-\sqrt{2} \), \( \sqrt{2} \).
- \( 2x^2 + 2 + 1 \). Find the roots with the quadratic formula. There will be three factors: one for each root and one for the coefficient 2 of \( x^2 \).
  \[
  2x^2 + 2x - 1
  \]
  Roots: \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4(2)(1)}}{2(2)} = \frac{-2 \pm \sqrt{4}}{4} = -1 \pm \frac{\sqrt{2}}{2}
  \]
  Factorization: \( 2(x - \frac{1 - \sqrt{2}}{2})(x - \frac{1 + \sqrt{2}}{2}) \)

Functions

DEFINITION. For sets \( A \) and \( B \), a function from \( A \) to \( B \) assigns a value \( f(x) \) in \( B \) to each \( x \) in \( A \). The domain of \( f \) is \( A \); the range of \( f \) is the set of all possible values \( f(x) \).

- \( f(x) = x^2 \) is a function from real numbers to real numbers.
  - domain = \((\infty, \infty) \) since \( x^2 \) is defined for all numbers.
  - range = \([0, \infty) \) since \( x^2 \) can never be negative.

NOTATION. Sometimes, instead of writing \( f(x) = x^2 \), we define a function by writing \( y = x^2 \).

Thus \( y \) is the value of the function. Since it depends on \( x \), \( y \) is the dependent variable. Since \( x \) ranges freely over the domain, it is the independent variable.

A function may assign only one value to each \( x \).
Thus \( y = \pm \sqrt{x} \) is not a function.

- Of \( f \) and \( g \), which are functions? (\( f \) isn’t, \( g \) is)

Write each domain in interval notation.
- \( y = 1 - x \quad (\infty, \infty) \)
- \( y = \frac{1}{1-x} \quad (\infty, 1) \cup (1, \infty) \)
- \( y = \sqrt{1-x} \quad (-\infty, 1] \)

- \( f(x) = x^2 \) First add ()’s around each \( x \): \((x)^2 \). Simplify to an expanded polynomial.
  - \( \frac{f(x) - f(a)}{x - a} = \frac{(x)^2 - (a)^2}{x - a} = x + a \)
  - \( \frac{f(x + h) - f(x)}{h} = \frac{(x + h)^2 - (x)^2}{h} = 2x + h \)

To get \( f(x + h) \), replace \( x \) in \( f(x) = x^2 \) by \( (x + h) \) to get \( f(x + h) = (x + h)^2 \).
Note, \( f(x + h) \neq x^2 + h^2 \).

- \( g(x) = \frac{1}{x} - x \). Rewrite as \( \frac{1}{x} - (x) \). Simplify
  \[
  g(g(x)) = \frac{1}{g(x)} - g(x) = \frac{1}{\left(\frac{1}{x} - x\right)} - \left(\frac{1}{x} - x\right)
  \]
  \[
  = \frac{1}{1-x^2} - \frac{1}{x} + x = \frac{x^2}{x(1-x^2)} - \frac{1-x^2}{x(1-x^2)} + \frac{x^2(1-x^2)}{x(1-x^2)} = \frac{x^4 + 3x^2 - 1}{x(1-x^2)}
  \]

- \( h(x) = \frac{1}{x} \). Simplify
  \[
  h(x + h) = \frac{1}{(x+h)} - \frac{1-x}{xh}
  \]
  \[
  h(h(x)) = \frac{1}{h(x)} - \frac{1}{\left(\frac{1}{x}\right)} = \frac{1-x}{xh} = \frac{x^2 - 1}{1-x}
  \]