Translations and Reflections

**Theorem.** For the graph of any function $f$,

- vertical changes $\leftrightarrow$ changes in the value $f(x)$,
- horizontal changes $\leftrightarrow$ changes in the argument $x$

and work in the opposite direction.

<table>
<thead>
<tr>
<th></th>
<th>reflect vertically around x-axis</th>
<th>reflect horizontally around y-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)+1$</td>
<td>$f(x)-1$</td>
<td>$-f(x)$</td>
</tr>
<tr>
<td>$f(x+1)$</td>
<td>$f(x-1)$</td>
<td>$f(-x)$</td>
</tr>
</tbody>
</table>

* Horizontal changes are the opposite of what one would expect.

Horizontal changes come from replacing $x$.
Replacements work in opposite the expected direction.

In $y = f(x)$ replacing $y$ by $y - 1$ gives

$y - 1 = f(x)$ which is the same as

$y = f(x) + 1$ which shifts up one unit.
Given $f(x) = |x|$

- $f(x) = |x|$,  
- $f(x) + 2 = |x| + 2$,  
- $f(x) - 2 = |x| - 2$,  
- $-f(x) = -|x|$

The value $f(x)$ = the height = the vertical position of a point on the graph.

Changing $f(x)$ changes the vertical position of the graph.

Adding 2 raises the graph 2 units.

Negating $f(x)$ reflects the graph vertically across the $x$-axis.
Horizontal Moves

Replacing \( x \) by \( x+2 \) shifts the graph \textbf{left} 2 units.

Replacing \( x \) by \( x-2 \) shifts the graph \textbf{right} 2 units.

Replacing \( x \) by \( -x \) reflects the graph horizontally across the \( y \)-axis.

\[
\begin{align*}
f(x) &= \sqrt{x}, & f(x+2) &= \sqrt{x+2}, & f(x-2) &= \sqrt{x-2}, & f(-x) \\
\end{align*}
\]
Given $f(x)$, find the graph of ---

- reflect in y-axis
- reflect in x and y-axis
- reflect in x-axis

- find the graph of $-f(x)$.
- find the graph of $f(-x)$.
- find the graph of $f(x - 2)$. 
Given $f(x)$, find the functions for the other graphs.

- **(1)** is the graph of

- **(2)** is the graph of
\[ f(x) = \frac{1}{x}. \] Graph \( f \) and describe the shifts.

\[ \frac{1}{x} \] is a hyperbola

\[ \frac{1}{x} + 1 \] Shift up 1

\[ -\frac{1}{x} \] Reflect vertically in \( x \)-axis

\[ \frac{1}{x - 1} \] Shift right 1

\[ \frac{1}{x + 1} \] Shift left 1

\[ \frac{1}{-x} \] Reflect horizontally across the \( y \)-axis
Every sequence of shifts and reflections of a function $f$, can be rewritten in the form

$$af(b(x - c)) + d$$

Apply these shifts and reflections in order from left to right.

A negative $a$ gives a vertical reflection.
A negative $b$ gives a horizontal reflection.
The horizontal shift is determined by $c$. Right shift for positive $c$; left shift for negative $c$.
The vertical shift is determined by $d$.

**Rewrite $-3 - f(-2 - x)$ in the above form and then list the sequence of shifts and reflections.**

$$-3 - f(-2 - x)$$
$$= -f(-x - 2) - 3$$
$$= -f(-(x + 2)) - 3$$
$$= -f(-(x - (-2))) + (-3)$$

The sequence of shifts and reflections is:
vertical reflection around $x$-axis,
horizontal reflection around the $y$-axis,
horizontal shift left 2,
vertical shift down 3.
Rewrite form: \( af(b(x - c)) + d \)

Find the sequence of shifts and/or reflections which carries \( f(x) \) to \( f(-x - 1) \).

\[ f(-x - 1) = f(-(x + 1)) = \ldots \]

\[ f(1 - x) \]
Rewrite form: $af(b(x-c)) + d$

$f(x) = \frac{1}{x}$.

Find the sequence of shifts and/or reflections needed to get

$$\frac{1}{x+1} + 1 = f(x + 1) + 1$$

Don’t just memorize steps, try to understand the math. If you understand, every test problem should be recognizable as equivalent to some problem you have done before. If don’t understand, there will be problems on the exam which you won’t recognize.
The graphs of $f$ and $g$ are

First rewrite it in the form $af(b(x-c)) + d$

Graph $1 - g(1-x)$

Graph $2 - f(2-x)$
Graph $2 - f(2 - x)$.

$-f(-(x - 2)) + 2$
Graph \( 1 - g(1 - x) \) ← Your problem.