Math 140  Lecture 26

CONVENTION. Assume side $a$ is opposite angle $A$, side $b$ is opposite angle $B$ and side $c$ is opposite angle $C$.

SINE LAWS. In any triangle, the ratio of one angle’s sine and its opposite side equals the ratio of any other angle’s sine and opposite side.

Although written as one, there are 3 equations. Each involves two sides and two angles.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

PROOF. Recall that the area of a triangle is half the product of any two sides times the sine of their included angle.

Thus the area of the triangle can be written three ways: 

$$\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$$

multiply by 2

$$bc \sin A = ac \sin B = ab \sin C$$

divide by $abc$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

COSINE LAWS. For any two sides of a triangle, the sum of their squares minus twice their product times the cos of the included angle equals the square of the third side.

Each involves three sides and one angle.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

PROOF. We prove the last equality for the case $C$ acute.

$$\sin C = \frac{h}{a}, \ \cos C = \frac{x}{a}, \ \text{so} \ h = a \sin C, \ x = a \cos C$$

$$c^2 = (b-x)^2 + h^2$$

$$c^2 = b^2 - 2bx + x^2 + h^2$$

$$c^2 = b^2 - 2b(a \cos C) + a^2 \cos^2 C + a^2 \sin^2 C$$

$$c^2 = b^2 - 2ab \cos C + a^2 (\cos^2 C + \sin^2 C)$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

STRAIGHT ANGLE SUM FACT. The sum of a triangle’s 3 angles is a straight angle.

$$\angle A + \angle B + \angle C = 180^\circ = \pi$$

If you know two angles, you can solve for the third. Solving for $C$ gives $\angle C = 180^\circ - (\angle A + \angle B)$.

Today’s problems involve 4 quantities; each is a side or an angle whose measure is given or wanted. Usually --

- For 2 sides and 2 angles: use the sine law involving the 2 sides (if necessary, get the third angle with the Straight Angle Sum Theorem).
- For 3 sides and 1 angle: use the cosine law involving the angle.

Given $b, \angle A, \angle C$: find $c$.

2 side, 2 angle problem. Use the sine law with $b$ and $c$.

$$\frac{c}{\sin C} = \frac{b}{\sin B} \quad \text{where} \ B = 180^\circ - (A+C)$$

Given $a, b, c$: find $\angle C$.

3 side, 1 angle problem. Use the cosine law with $\angle C$.

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Solve for $\cos C$ then take $\cos^{-1}$.

Give an exact answer and a 2-place decimal answer.

$\angle A = 20^\circ, \ \angle B = 30^\circ, \ c = 40\text{cm}$. Find $a$.

2 angle, 2 side problem with sides $a, c$.

Use the sine law for $c$ and $a$.

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \text{where} \ C = 180^\circ - (\angle A + \angle B)$$

$$= 180^\circ - (20^\circ + 30^\circ) = 180^\circ - 50^\circ = 130^\circ$$

Now solve for $a$.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$a = \frac{c \sin A}{\sin C} = \frac{40 \sin 20^\circ}{\sin 130^\circ} \text{ cm} \quad \leftarrow \text{Exact answer}$$

$$= 17.86 \text{ cm} \leftarrow \text{2-place decimal answer}$$

Additional table-user step:

$$\sin x = \sin (\pi - x) = \sin (180^\circ - x), \ \text{so}$$

$$\sin C = \sin 130^\circ = \sin (180^\circ - 130^\circ) = \sin 50^\circ$$

$\angle A = 20^\circ, \ b = 50, \ c = 60$. Find $a$.

3 side, 1 angle problem with angle $\angle A$.

Use the cosine law for $\angle A$.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Solve for $a$.

$$a = \sqrt{b^2 + c^2 - 2bc \cos A}$$

$$= \sqrt{50^2 + 60^2 - 2(50)(60) \cos 20^\circ}$$

$$= \sqrt{6100 - 6000 \cos 20^\circ} \quad \leftarrow \text{Exact answer}$$

$$= 21.49 \quad \leftarrow \text{2-place decimal answer}$$

$\angle A = 20^\circ, \ b = 20, \ c = 30$. Find $\angle A$.

3 side, 1 angle problem with angle $\angle A$.

Use the cosine law for $\angle A$.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Solve for $\cos A$.

$$2bc \cos A = b^2 + c^2 - a^2$$

$$\cos A = \frac{(b^2 + c^2 - a^2)}{2bc}$$

$$A = \cos^{-1} [(b^2 + c^2 - a^2) / 2bc]$$

$$= \cos^{-1} (900/1200) = \cos^{-1} (3/4) = 41.41^\circ$$