Math 140  Lecture 27

A triangle is determined uniquely up to congruence (1) by two sides and an included angle, and also, (2) by two angles and an included side.

However, given two sides and a nonincluded angle, there may be 0, 1, or 2 triangles.

Suppose \( \angle A = 30^\circ, b = 2 \). Then there is no triangle with \( a = .5, \) one triangle with \( a = 1, \) two triangles with \( a = 1.5, \) and one triangle with \( a = 2.5, \)

```
\( \begin{array}{c|c}
\text{a} & \text{b} \\
\hline
5 & 2 \\
1 & 2 \\
1.5 & 2 \\
2.5 & 2 \\
\end{array} \)
```

Suppose two sides and a nonincluded angle are known.

When solving for the third side using a cosine law, you may get an answer of the form \( s = \pm \sqrt{t} \). Then there is:

- No solution if \( t < 0 \) or both \( s = \pm \sqrt{t} < 0 \).
- Two solutions if \( t > 0 \) and both \( s = \pm \sqrt{t} > 0 \).
- One solution otherwise.

When solving for a second angle using a sine law, you may get an answer of the form \( \theta = \sin^{-1}t, t \geq 0 \). There is:

- No solution if \( t > 1 \).
- Two solutions if \( t < 1 \) and the larger of the two sides is adjacent to the given angle (\( \theta \) is one angle, \( \pi - \theta \) the other).
- One solution otherwise.

In a triangle, \( \sin A = 1/2 \). What are the possible angles, in degrees, for \( A \)? One is \( A = 30^\circ \), the other is \( 180^\circ - 30^\circ = 150^\circ \).

Try doing this and the next problem by drawing accurate pictures.

Is there a triangle in which \( a = 2, b = 3, \) and \( \angle A = 60^\circ \)?

Such a triangle exists if the third side \( c \) exists.

Solving for \( c \).

\[
a^2 = b^2 + c^2 - 2bc \cos A \\
4 = 9 + c^2 - 2(3)c \cos 60^\circ \\
0 = c^2 - 2(3)c \cdot \frac{1}{2} + 5 \\
c^2 - 3c + 5 = 0 \\
c = \frac{3 \pm \sqrt{3^2 - 4(1)(5)}}{2(1)} = \frac{3 \pm \sqrt{-11}}{2} = \text{undefined}
\]

Since \( c \) is undefined, the triangle does not exist.

\( a = 2, b = 2, \angle A = 30^\circ \). Find \( \angle C \) if \( \angle B \) is acute.

First find \( \angle B \) using the sine law, then find \( \angle C \).

\[
\frac{\sin B}{b} = \frac{\sin A}{a}, \quad \sin B = \frac{b}{a} \sin A. \quad \text{Thus one answer is} \quad B = \sin^{-1}(\frac{b}{a} \sin A). \quad \text{The other answer is} \quad 180^\circ - 30^\circ = 150^\circ. \quad \text{The acute angle is} \quad 30^\circ. \quad \angle C = 180^\circ - (\angle A + \angle B) = 180^\circ - (30^\circ + 30^\circ) = 120^\circ.
\]

### Polar coordinates

**Definition.** The **polar coordinates** \((r, \theta)\) of a point \( P \) are its distance \( r \) (radius) from the origin and the angle \( \theta \) between the positive x-axis and the line from (0,0) to \( P \). The usual \((x, y)\) are the **rectangular coordinates**.

- **Plot the polar coordinates points** \((1, \pi/4), (2, -\pi/2)\).
- From the picture we have, for positive \( r \)
  \[
  \sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}, \quad r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}
  \]
  \[
  y = r \sin \theta, \quad x = r \cos \theta, \quad r = \sqrt{x^2 + y^2}
  \]
  \[
  \theta = \tan^{-1} \frac{y}{x} \quad \text{if} \; \theta \in I, IV \quad \theta = \tan^{-1} \frac{y}{x} + \pi \quad \text{if} \; \theta \in II, III
  \]

**Theorem.** If a point has rectangular coordinates \((x, y)\) and polar coordinates \((r, \theta)\) with \( r \geq 0 \), then:

\( (x, y) = (r \cos \theta, r \sin \theta) \) and

\( (r, \theta) = (\sqrt{x^2 + y^2}, \arctan \frac{y}{x}) \) if \((x, y)\) is in quadrant I, IV,

\( (r, \theta) = (\sqrt{x^2 + y^2}, \arctan \frac{y}{x} + \pi) \) if \((x, y)\) in quadrant II, III.

**Negative** \( r, -r, \theta \) is the point \((r, \theta + \pi)\) on the opposite side of the origin \((0,0)\) as \((r, \theta)\).

- **Convert from polar coordinates to rectangular:** \((7, \frac{\pi}{3})\).
  
  \[
  r = 7, \quad \theta = \pi/6, \quad (r \cos \theta, r \sin \theta) = (7 \cos \frac{\pi}{6}, 7 \sin \frac{\pi}{6}) = (\frac{7 \sqrt{3}}{2}, \frac{7}{2}) \quad \Rightarrow \text{answer}
  \]

- **Convert from rectangular coordinates to polar:**\((-1, \sqrt{3})\).
  
  \[
  x = -1, \quad y = \sqrt{3}. \quad \text{Point is in quadrant II,} \; : \; \text{add} \; \pi.
  \]
  \[
  (r, \theta) = (\sqrt{x^2 + y^2}, \pi + \tan^{-1} \frac{y}{x}) = (\sqrt{(-1)^2 + \sqrt{3}^2}, \pi + \frac{-\sqrt{3}}{1}) = (2, \pi - \frac{\pi}{3}) = (2, \frac{2\pi}{3}) \quad \Rightarrow \text{answer}
  \]

- **Convert the polar equation to a rectangular equation:**
  
  \[
  r \sin \theta + 2 \cos \theta = 0 \\
  y + 2(x/r) = 0 \Rightarrow y + 2 \frac{x}{\sqrt{x^2+y^2}} = 0 \]
  \[
  y \sqrt{x^2 + y^2} + 2x = 0 \quad \Rightarrow \text{answer (hw simplifies more)}
  \]

- **Convert the rectangular equation to a polar equation:**
  
  \[
  2x - y^2 = 0 \\
  2r \cos \theta - r^2 \sin^2 \theta = 0 \quad \Rightarrow \text{answer}
  \]