Solving quadratic equations

Find all real numbers $x$ such that $9x + 12 = -\frac{5}{x}$.
First rewrite this with everything on the left, 0 on the right.
\[9x + 12 = -\frac{5}{x}\]
\[9x + 12 + \frac{5}{x} = 0\]
\[9x^2 + 12x + 5 = 0\]

First try factoring. In this case, it doesn’t work, so use the quadratic formula:
\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]
\[x = \frac{-12 \pm \sqrt{144 - 180}}{18} = \text{undefined}\]

Since 144-180 is clearly negative, we can’t take the square root, hence there are no solutions. Don’t use the imaginary number $i$ here, the problem asks for real numbers.

Answer: no solution.

Find all real numbers $b$ such that $\sqrt{b} = b$.

Square to get rid of the square root, then put everything on the left.
\[\sqrt{b} = b\]
\[b = b^2\]
\[b - b^2 = 0\]
\[b^2 - b = 0\]
We multiply by -1 and put the highest power first to make things look more familiar.

Now factor.
\[b(b - 1) = 0\]

Set each factor to zero separately.

Answer: $b = 0, 1$

Find all real numbers $d$ such that $6d^2 = 5d - 2$.
First rewrite this with everything on the left, 0 on the right.
\[5d^2 - 7d + 2 = 0\]

Now try to factor.
\[(5d - 2)(d - 1) = 0\]

Set each factor to zero separately.

Answer: $d = \frac{2}{5}, 1$

Completing the square

Complete the square for $2x^2 - 7x + 12$.

Warning: completing the square for the formula $2x^2 - 7x + 12$ and completing the square for the equation $2x^2 - 7x + 12 = 0$ are two different problems. The common mistake is to assume the formula is an equation.

For the equational version of completing the square, see the circle equation section of Lecture 2. Completing the square for formulas is different. With an equation, you can add a number to both sides. When you add something to a formula, you must subtract it off somewhere else in the formula.

\[2x^2 - 7x + 12\]

First factor the coefficient of $x^2$ out of the first two terms. Leave the constant alone. Factoring a 2 out of 7x is the same as dividing 7x by 2.
\[(2x^2 - 7x) + 12\]
\[2(x^2 - \frac{7}{2}x) + 12\]

To complete the square of a factor $(x^2 + ax)$, divide the coefficient of $x$ by two, then square. This gives $(\frac{a}{2})^2$.

We add this to the factor in parentheses to get $(x^2 + ax + (\frac{a}{2})^2)$ which is the perfect square $(x + \frac{a}{2})^2$.

In $(x^2 - \frac{7}{2}x)$, $a = -\frac{7}{2}$.

Dividing by 2 and squaring gives: $-\frac{7}{2} \rightarrow -\frac{7}{4} \rightarrow \frac{49}{16}$.

Thus $2(x^2 - \frac{7}{2}x) + 12$ becomes
\[2(x^2 - \frac{7}{2}x + \frac{49}{16}) + 12 - 2(\frac{49}{16})\]

Since we added $\frac{49}{16}$, we compensate by subtracting it from the end of the formula. Where did the 2 come from?
Since the $\frac{49}{16}$ that we added was multiplied by 2 (the 2 at the beginning of the formula), the $\frac{49}{16}$ we subtract must also be multiplied by 2. Failure to do this multiplication is a common error.

Hint. Cancellation almost always occurs at this step.
Don’t multiply $2 \times 49$; cancel 2 off the 16. You should get $\frac{49}{8}$ rather than $\frac{98}{16}$.
\[2(x - \frac{7}{4})^2 + \frac{49}{8} - \frac{49}{8}\]
\[2(x - \frac{7}{4})^2 + \frac{47}{8}\]

Answer: $2(x - \frac{7}{4})^2 + \frac{47}{8}$

Be careful when adding fractions; partial credit isn’t allowed on gateway exams.
Inequalities and absolute values

You can replace absolute value inequalities with a pair of algebraic inequalities.

\[
|a| > 0 \text{ iff } a \neq 0 \\
|a| > 2 \text{ iff } a < -2 \text{ or } 2 < a \\
|a| < 2 \text{ iff } 0 - 2 < a < 2
\]

- Use inequalities to write \(|z + 1| > 0\) without absolute value signs.
  \[|z + 1| > 0 \text{ iff } z + 1 \neq 0 \text{ iff } z \neq -1\]
  Answer: \(z \neq -1\)

- Use inequalities to write \(|z + 3| \geq 3\) without absolute value signs.
  \[|z + 3| \geq 3 \text{ iff } z + 3 \leq -3 \text{ or } 3 \leq z + 3\]
  \[\iff z \leq -6 \text{ or } 0 \leq z\]
  Answer: \(z \leq -6\) or \(z \geq 0\)

- Use inequalities to write \(|t + \frac{7}{2}| < \frac{3}{2}\) without absolute value signs.
  \[|t + \frac{7}{2}| < \frac{3}{2} \iff -\frac{3}{2} < t + \frac{7}{2} < \frac{3}{2}\]
  \[\iff -5 < t < -2\]
  Answer: \(-5 < t < -2\)

Express the following inequalities by using one pair of absolute value signs:

\[t \leq 1 \text{ or } 4 \leq t\]

Draw a picture of the set; find the midpoint between the endpoints; find the distance between the midpoint and the endpoints.

The distance between \(a\) and \(b\) is \(|a - b|\).
The midpoint between \(a\) and \(b\) is \((a + b)/2\).

Picture:

\[\begin{array}{ccc}
  & 1 & 5/2 & 4 \\
  & 3/2 & 3/2 & \\
\end{array}\]

The midpoint is \((1+4)/2 = 5/2\).
The distance between the midpoint and the endpoints is \(5/2 - 1 = 3/2\).
\(t\) is in the set
\iff the distance between \(t\) and \(5/2\) is greater than \(3/2\).
Answer: \(|t - \frac{5}{2}| \geq \frac{3}{2}\)

Domain problems

- Let \(g(d) = \sqrt{1/d}\). State the domain of \(g\) and compute \(g(1/16)\).
  
  You can’t divide by 0, you can’t take the square root of a negative.
  Hence we must have \(d \neq 0\) and we must have \(1/d \geq 0\).
  That latter means \(d \geq 0\). This plus \(d \neq 0\) gives
  
  Answer: domain: \(d > 0\).

  \[g(1/16) = \sqrt{1/(1/16)} = \sqrt{16} = 4 = 2\]
  
  Answer: \(g(1/16) = 2\).

- Let \(g(x) = x/x\). What is the domain of \(g\) and what is \(g(2)\)?
  
  You can’t divide by 0. Hence we must have \(x \neq 0\).
  Answer: domain: \(x \neq 0\).
  
  Note: Domains must be calculated without simplifications. If we simplified \(x/x\) to 1, we would wrongly conclude that the domain was \((-\infty, \infty)\).

  \[g(2) = 2/2 = 1\]
  
  Answer: \(g(2) = 1\).
Exponential equation: variable in exponent

Rewrite the equation so that both sides are powers of the same base $b^p = b^q$. Then equate the exponents $p = q$ and solve for the variable.

■ Solve for $n$: \( \frac{2^{5/3}}{4^{1/3}} = 2^n \).
Since $4 = 2^2$, everything can be written as a power of 2.

\[
\frac{2^{5/3}}{2^{2/3}} = 2^n
\]
\[
2^{5/3 - 2/3} = 2^n \quad \text{You don’t have to do this much detail.}
\]
\[
2^{(5/3 - 14/3)} = 2^n \quad \text{If you can do this in your head, great.}
\]
\[
2^{-9/3} = 2^n
\]
\[
2^{-3} = 2^n
\]
\[
-3 = n
\]
Answer: $n = -3$

■ Solve for $x$: \( 24^{2/3} \cdot 3^{-2/3} = 8^{-x+1} \).
First simplify.

\[
24^{2/3} \cdot 3^{-2/3} = 8^{-x+1}
\]
\[
(3 \cdot 8)^{2/3} \cdot 3^{-2/3} = 8^{-x+1}
\]
\[
3^{2/3} \cdot 8^{2/3} \cdot 3^{-2/3} = 8^{-x+1} \quad \text{since } 3^{2/3} \cdot 3^{-2/3} = 3^{(2/3 - 2/3)} = 3^0 = 1
\]
\[
8^{2/3} = 8^{-x+1} \quad \text{Now equate exponents.}
\]
\[
x + 2/3 = 1
\]
\[
x = 1 - 2/3 = 1/3
\]
Answer: $x = \frac{1}{3}$

Exponential equation: variable in base

■ Solve for $x$: \( \frac{6^{3/2}}{(\sqrt[3]{2})^3} = x^{1/2} \).
To get $x$, square both sides.

\[
(\frac{6^{3/2}}{(\sqrt[3]{2})^3})^2 = (x^{1/2})^2
\]
\[
\frac{(6^{3/2})^2}{(\sqrt[3]{2})^6} = x
\]
\[
x = \frac{6^3}{(2^{1/2})^6}
\]
\[
x = \frac{6^3}{2^3} = \frac{(2 \cdot 3)^3}{2^3} = \frac{2^3 \cdot 3^3}{2^3} = 3^3 = 27
\]
Answer: $x = 27$

■ Solve for $x$: \( \frac{36}{9^{3/2}} = x^{33} \).
To get $x$, take the $33$rd root of both sides.

\[
x^{33} = \frac{36}{9^{3/2}}
\]
\[
(x^{33})^{1/33} = \left(\frac{36}{9^{3/2}}\right)^{1/33}
\]
\[
(x^{33})^{1/33} = \left(\frac{36}{(3^2)^{3/2}}\right)^{1/33}
\]
\[
x = \left(\frac{36}{3^{6/2}}\right)^{1/33} = (36 \cdot 3^5)^{1/33}
\]
\[
x = (3^{11})^{1/33} = 3^{11/33} = 3^{1/3}
\]
Answer: $\sqrt[3]{3}$