Math 140  Lecture 3  

Reminder: Gateway Exam week from Thursday.

Factoring and roots

**Theorem.** If \( a > 0 \), \( x^2 - a = (x - \sqrt{a})(x + \sqrt{a}) \). But \( x^2 + a \) has no roots and can’t be factored any more.

**Division Law.** If \( p(x)/d(x) \) has quotient \( q(x) \) and remainder \( r(x) \) then \( \frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)} \). Multiply by \( d(x) \) to get \( p(x) = d(x)q(x) + r(x) \).

d(x) divides into \( p(x) \) evenly iff the remainder is 0 iff \( p(x) = d(x)q(x) \) iff \( d(x) \) is a factor of \( p(x) \).

If \( d(x) \) is a factor of \( p(x) \), the other factor of \( p(x) \) is the quotient factor \( q(x) \). To get this quotient factor divide: \( p(x)/d(x) \).

- Given \( p(x)/d(x) \), divide to get the quotient \( q(x) \) and remainder \( r(x) \). Write the answer in division law form: \( p(x) = d(x)q(x) + r(x) \).
  
  \[
  \begin{align*}
  x^3 + 1, & \quad \frac{x^3 + 1}{x - 1} = x^2 + x + 1 + \frac{2}{x-1}, \quad x^3 + 1 = (x - 1)(x^2 + x + 1) + 2 \\
  \text{Check that the answer is correct for } x = 0. \\
  \text{For } x = 0, \text{ we get } 0 + 1 = (-1)(0 + 0 + 1) + 2, 1 = 1. 
  \end{align*}
  \]

**X-Intercept.** \( a \) is a root or zero of \( p(x) \) iff \( p(a) = 0 \).

**Theorem.** \( a \) is a root of \( p(x) \) iff \( (x-a) \) is a factor of \( p(x) \).

To find all roots of \( p(x) \), completely factor \( p(x) \).

*Factor the polynomial and find all roots.*

- \( x + 2 \) \hspace{1cm} Root: -2
- \( x^2 + 2 \) \hspace{1cm} Fully factored as is, no roots.
- \( x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2}) \) \hspace{1cm} Roots: \( -\sqrt{2}, \sqrt{2} \)
- \( x^2 - 4x + 4 = (x - 2)^2 \) \hspace{1cm} One repeated factor. Root: 2
- \( x^2 + 5x^2 + 8x + 4 \) \hspace{1cm} Given that \(-1\) is a root.
  
  \[(x - a) = (x - (-1)) = (x + 1) \quad \therefore \quad \text{we divide by } (x + 1).
  \]
  
  \[
  \begin{align*}
  \frac{x^3 + 5x^2 + 8x + 4}{x + 1} &= x^2 + 4x + 4 = (x + 2)(x + 2) \\
  x^3 + 5x^2 + 8x + 4 &= (x + 1)(x + 2)^2 \\
  \text{Roots: } -1, -2. 
  \end{align*}
  \]
- \( x^3 - x^2 - 2x + 2 \) \hspace{1cm} Given that 1 is a root.
  
  \[
  \begin{align*}
  \frac{x^3 - x^2 - 2x + 2}{x - 1} &= x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2}) \\
  x^3 - x^2 - 2x + 2 &= (x - 1)(x + \sqrt{2})(x - \sqrt{2}) \\
  \text{Roots: } -\sqrt{2}, \sqrt{2}. 
  \end{align*}
  \]
- \( 2x^2 + 2x - 2. \) Factor out the coefficient of \( x^2 \); find the roots with the quadratic formula; factor.
  
  \[
  2(x^2 + x - 1)
  \]
  
  **Roots:** \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}, \frac{-1 - \sqrt{5}}{2} \)
  
  **Factorization:** \( 2(x - \frac{-1 + \sqrt{5}}{2})(x - \frac{-1 - \sqrt{5}}{2}) \)

**Functions**

**Definition.** For sets \( A \) and \( B \), a function from \( A \) to \( B \) assigns a value \( f(x) \) in \( B \) to each \( x \) in \( A \). The domain of \( f \) is \( A \); the range of \( f \) is the set of all possible values \( f(x) \).

- \( f(x) = x^2 \) is a function from real numbers to real numbers.
  
  \[
  \text{Domain } = (-\infty, \infty) \quad \text{since } x^2 \text{ is defined for all numbers.}
  \]
  
  \[
  \text{Range } = [0, \infty) \quad \text{since } x^3 \text{ can never be negative.}
  \]

**Notation.** Sometimes, instead of writing \( f(x) = x^2 \), we define a function by writing \( y = x^2 \).

Thus \( y \) is the value of the function. Since it depends on \( x \), \( y \) is the dependent variable. Since \( x \) ranges freely over the domain, it is the independent variable.

A function may assign only one value to each \( x \).

Thus \( y = \pm \sqrt{x} \) is not a function.

- Of \( f \) and \( g \), which are functions? \( (f \text{ isn’t, } g \text{ is) \n
  \begin{align*}
  f(2) &= 4 \\
  f(x)^2 + 2 &= x^2 + 2 \\
  xf(x) &= x^3 \\
  f(x^2) &= x^6 \\
  (f(x))^3 &= x^6 \\
  f(f(x)) &= x^4 \\
  \frac{f(x) - f(a)}{x - a} &= \frac{x^2 - a^2}{x - a} = x + a \\
  \frac{f(x + h) - f(x)}{h} &= \frac{(x + h)^2 - x^2}{h} = 2x + h 
  \end{align*}
  \]