Math 140  Lecture 6
Study Practice Exam 1 and the recommended exercises.

Functions can be added and multiplied just like numbers.

**Definition.** For functions \( f \) and \( g \), define \( f+g, f-g, fg, f/g \) by
\[
(f+g)(x) = f(x) + g(x), \\
(f-g)(x) = f(x) - g(x), \\
(fg)(x) = f(x)g(x), \\
(f/g)(x) = f(x)/g(x).
\]
Note: \((f+g)(x)\) is not \((f+g)(x)\)
The first is function application.
The second is multiplication.

If \( f(x) = x - 2 \), \( g(x) = 6 \), then
\[
(f+g)(x) = f(x) + g(x) = (x-2) + (6) = x + 4, \\
(f-g)(x) = f(x) - g(x) = (x-2) - (6) = x - 8, \\
(fg)(x) = f(x)g(x) = (x-2)(6) = 6x - 12, \\
(f/g)(x) = f(x)/g(x) = (x-2)/6 = \frac{1}{6}x - \frac{1}{3}.
\]

**Definition.** For functions \( f \) and \( g \), define \( f \circ g \), the composition of \( f \) and \( g \), by
\[
(f \circ g)(x) = f(g(x))
\]
Apply \( g \) to \( x \). Get \( g(x) \). Apply \( f \) to \( g(x) \). Get \( f(g(x)) \). 
\( f \) is the outer function; \( g \) is the inner function.

Suppose \( f(x) = x - 2 \) and \( g(x) = x^2 \).
\[(a) \text{ Find } f \circ g \text{ and } g \circ f.
\]
\[
(f \circ g)(x) = f(g(x)) = f(x^2) = x^2 - 2, \\
g(\circ f)(x) = g(f(x)) = g(x-2) = (x-2)^2 = x^2 - 4x + 4.
\]
Note that \( f \circ g \neq g \circ f \). For composition, order matters.

\[(b) \text{ Find } (f \circ g)(2). 
\]
Since \((f \circ g)(x) = x^2 - 2 \), \((f \circ g)(2) = 2^2 - 2 = 4 - 2 = 2 \).

\[(b') \text{ Find } (f \circ g)(2) \text{ and } (g \circ f)(2) \text{ directly without } (f \circ g)(2), 
\]
\[
(g \circ f)(x) = g(f(x)) = g(x-2) = g(2) = 0 \text{ if } f(x) = 2.
\]
\[(b'') \text{ Find } (f \circ g)(2) \text{ and } (g \circ f)(2) \text{ directly without } (f \circ g)(x), 
\]
\[
(f \circ g)(2) = f(g(2)) = f(2) = 4 - 2 = 2, \\
g(\circ f)(2) = g(f(2)) = g(2) = g(0) = 0 \text{ if } g(x) = 2.
\]

If \( h(x) = c \), then \( h(8) = c \), \( h(y) = c \), \( h(x^2 - 1) = c \), \( h(g(x)) = c \).

For \( f(x) = 3x + 4 \), \( g(x) = 5 \), find \( f \circ g \) and \( g \circ f \).
\[
f(g(x)) = f(5) = 3 \cdot 5 = 15 + 4 = 19, \\
g(f(x)) = g(3x + 4) = 5.
\]

For \( f \) and \( g \) above, note that
\[
f(-3) = 1, f(-1) = 2, f(2) = 3, f(4) = 2, \\
g(-2) = -1, g(-1) = -2, g(1) = -1, g(2) = 2.
\]

Find
\[
f(\circ g)(2) = f(g(2)) = f(2) = 3, \\
g(\circ f)(2) = g(f(2)) = g(3) =unday, \\
f(\circ f)(1) = f(f(1)) = f(2) = 3.
\]
\[
f(x) = x + \frac{1}{x}, f(f(x)) = (x + \frac{1}{x}) + 1/(x + \frac{1}{x}) = x + \frac{1}{x} + \frac{x}{x^2 + 1}
\]

Write each function below as a composition
\( f (g(x)) \) of two simpler functions, an outer function \( f \) and an inner function \( g \).

Find the inner function first.

\[\text{Write } (x^2 + 2)^6 \text{ as a composition } f(g(x)). \]
\[x^2 + 2 \text{ inner function } g(x) = x^2 + 2 \]
outer function \( f(x) \) does what remains to be done. \( \therefore \)
\[
f(x) = x^6, \\
check: f(g(x)) = f(x^2 + 2) = (x^2 + 2)^6.
\]

\[\text{Write } 4x^2 + 3 \text{ as a composition } f(g(x)). \]
\[4x^2 + 3 \text{ inner function } g(x) = \frac{1}{x} \]
outer function \( f(x) \) does what remains to be done. \( \therefore \)
\[
f(x) = 4x + 3, \\
check: f(g(x)) = f(\frac{1}{x}) = 4(\frac{1}{x}) + 3.
\]

\[\text{Write } \sqrt{x + 1} \text{ as a composition.} \]
\[\sqrt{x + 1} \]
\[
g(x) = x + 1, \\
f(x) = \sqrt{x} \]
\[
\therefore f(g(x)) = f(x + 1) = \sqrt{x + 1}.
\]

\[\text{Write } x^4 + x^2 + 1 \text{ as a composition.} \]
\[x^4 + x^2 + 1 = (x^2)^2 + (x^2) + 1. \\
g(x) = x^2, \\
f(x) = x^2 + x + 1 \]
\[
\therefore f(g(x)) = f(x^2) = x^4 + x^2 + 1.
\]

\[\text{Write } \sqrt{x}/(1 + \sqrt{x}) \text{ as a composition of 2 functions.} \]
\[\sqrt{x}/(1 + \sqrt{x}) \]
Write \( 1/(1 + \sqrt{x}) \) as a composition of 3 functions.
\[\text{=h(f(g(x))), } g(x) = \sqrt{x}, f(x) = 1 + x, h(x) = 1/x \]

**Definition.** \( id(x) = x \) is called the identity function.

\[\text{Hence } id(5) = 5, \text{ } id(y) = y, \text{ } id(x^2 - 1) = x^2 - 1, \ldots . \]

**Theorem.** For any function \( f(x) \), \( f \circ id = f \) and \( id \circ f = f \).

**Proof.** \( f \circ id(x) = f(id(x)) = f(x). \)
\[ (id \circ f)(x) = id(f(x)) = f(x). \]
\[ 0 \text{ is the identity for addition, since } f + 0 = 0 + f = f. \]
\[ 1 \text{ is the identity for multiplication, } f \cdot 1 = 1 \cdot f = f. \]
\[ id(x) \text{ is the identity for composition, since } \]
\[ f \circ id = id \circ f = f. \]

\[ f(\circ g)(2) = f(g(2)) = f(2) = 3, \\
g(\circ f)(2) = g(f(2)) = g(3) = undefined, \\
f(\circ f)(1) = f(f(1)) = f(2) = 3.
\]
\[
f(x) = x + \frac{1}{x}, f(f(x)) = (x + \frac{1}{x}) + 1/(x + \frac{1}{x}) = x + \frac{1}{x} + \frac{x}{x^2 + 1}
\]