Math 140    Lecture 7

Inverse functions

**Definition.** \( f^{-1} \), the inverse of \( f \), is the function, if any, such that,
\[
\begin{align*}
    f^{-1}(f(x)) &= x & \text{when } f^{-1}(x) \text{ is defined and} \\
    f(f^{-1}(x)) &= x & \text{when } f(x) \text{ is defined.}
\end{align*}
\]

This says that \( f \) and \( f^{-1} \) undo each other:
\[
    f^{-1} \text{ undoes what } f \text{ does and gives you back } x.
\]

- \( f(x)=2x, \quad f^{-1}(x)=\frac{1}{2}x. \) Verify: \( f(f^{-1}(x))=x \) & \( f^{-1}(f(x))=x \)
- \( g(x)=x+3, \quad g^{-1}(x)=x-3. \)
- \( h(x)=2x+3, \quad h^{-1}(x)=(x-3)/2=\frac{1}{2}x-\frac{3}{2}. \)

Verify that \( h^{-1}(h(x))=x. \)

To undo a sequence of operations, you must undo them in the reverse order: the inverse of \( g(f(x)) \) is \( f^{-1}(g^{-1}(x)) \).

Let \( y=f^{-1}(x) \).
\[
f(f^{-1}(x))=x, \quad \text{by definition of inverse.} \]
\[
\therefore f(y)=x, \quad \text{since } y=f^{-1}(x).
\]
The converse is also true, thus

**Theorem.** \( y=f^{-1}(x) \) iff \( f(y)=x. \)

To find \( f^{-1}(x) \) for complicated functions:

Start with \( f(y)=x. \)

Solve for \( y \) to get \( y=f^{-1}(x) \).

- \( f(x)=x^3 \), find \( f^{-1}(x) \).
\[
f(y)=x  \\
\therefore y^3=x  \\
\therefore y=\sqrt[3]{x}. \quad f^{-1}(x)=\sqrt[3]{x}
\]

**Warning.** \( f^{-1}(x) \) and \( (f(x))^{-1} \) are not the same.
\[
f^{-1}(x) = \text{the inverse}  \\
(f(x))^{-1} = \text{the reciprocal} = 1/f(x).
\]

If \( f(x)=x^3 \)
\[
f^{-1}(x)=\sqrt[3]{x} \quad f^{-1}(0)=0  \\
(f(x))^{-1}=1/x^3 \quad (f(0))^{-1}=\text{undefined}.
\]

- \( f(x)=\frac{x+1}{x-1} \), find \( f^{-1}(x) \).
\[
f(y)=x, \quad \frac{y+1}{y-1}=x, \quad y+1=x(y-1)  \\
y+1=xy-x, \quad y-xy=-x-1  \\
y(1-x)=-x-1, \quad y=\frac{x+1}{1-x} = \frac{x+1}{x-1}  \\
\therefore f^{-1}(x)=\frac{x+1}{x-1}
\]

- If \( f(x)=x+3 \) then \( f^{-1}(x)=x-3. \)
- If \( g(x)=x/2 \) then \( g^{-1}(x)=2x. \)
- If \( h(x)=\sqrt{x} \) then \( h^{-1}(x)=x^2 \) for \( x \geq 0. \)

Note how the graph of \( f \) is related to the graph of \( f^{-1} \).

By the Theorem, \( y=f^{-1}(x) \) iff \( x=f(y) \). Thus the graph of \( y=f^{-1}(x) \) is the graph of \( f(y)=x \) which is just the graph of \( f(x)=y \) with \( x \) and \( y \) interchanged. Interchanging \( x \) and \( y \) reflects the plane around the major diagonal \( y=x \). Hence

**Theorem.** The graph of \( y=f^{-1}(x) \) is the reflection of the graph of \( y=f(x) \) across the major diagonal \( y=x \).

For each function, draw the three graphs \( y=f(x) \), \( y=x \), \( y=f^{-1}(x) \) on the same coordinate system.

- \( f(x)=x^3 \)
- \( f(x)=-x^3 \)

**Definition.** \( f \) is 1-1 (“one-to-one”) iff \( x \neq y \Rightarrow f(x) \neq f(y) \).

- \( f(x)=3x \text{ is 1-1} \quad x \neq y \Rightarrow 3x \neq 3y \)
- \( f(x)=x^2 \text{ is not} \quad 1 \neq 1 \text{ but } (-1)^2 = 1^2. \)

**Theorem.** The following are equivalent:
- \( f \) has an inverse
- \( f \) is 1-1
- no horizontal line intersects its graph more than once.

- Which of the following functions has an inverse?

![](image)

**Theorem.** The domain of \( f^{-1} \) is the range of \( f \). The range of \( f^{-1} \) is the domain of \( f \).

**Proof.** The reflection around the major diagonal which carries the graph of \( f \) to the graph of \( f^{-1} \) also carries the domain of \( f \) to the range of \( f^{-1} \) and the range of \( f \) to the domain of \( f^{-1} \).

Stated in full, the inverse is the compositional inverse.

Compare it with the additive inverse and the multiplicative inverse.

For addition, 0 is the identity and the additive inverse of \( f \) is the negative \(-f \) since \( f + (-f) = 0 \).

For multiplication, 1 is the identity and the multiplicative inverse of \( f \) is the reciprocal \( 1/f \) since:
\[
f \cdot (1/f) = (1/f) \cdot f = 1.
\]

For composition, \( id \) is the identity where \( id(x)=x \) and for the inverse \( f^{-1} \) of \( f \), the corresponding equation is \( f \circ f^{-1} = f^{-1} \circ f = id \).