Math 140     Lecture 13

Recall: Since \( \log_b x \) and \( b^x \) are inverses of each other:
- \( \log_b x = y \) iff \( x = b^y \)
- \( \log_b b^x = x \)
- \( b^{\log_b x} = x \)

Most properties of logarithms are the same as for exponentiation but with

<table>
<thead>
<tr>
<th>the symbols</th>
<th>0</th>
<th>+</th>
<th>−</th>
<th>( n(__) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>exchanged with</td>
<td>1</td>
<td>×</td>
<td>÷</td>
<td>( (__)^n )</td>
</tr>
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</table>

Assume \( b > 0, b \neq 1 \). On the log side, assume \( x, y > 0 \).

**LOG PROPERTIES**

<table>
<thead>
<tr>
<th>log properties</th>
<th>EXPONENT PROPERTIES</th>
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<tr>
<td>( \log_b 1 = 0 )</td>
<td>( b^0 = 1 )</td>
</tr>
<tr>
<td>( \log_b b = 1 )</td>
<td>( b^1 = b )</td>
</tr>
<tr>
<td>( \log_b xy = \log_b x + \log_b y )</td>
<td>( b^{x+y} = b^x b^y )</td>
</tr>
<tr>
<td>( \log_b x/y = \log_b x - \log_b y )</td>
<td>( b^{x-y} = b^x/b^y )</td>
</tr>
<tr>
<td>( \log_b x^n = n \log_b x )</td>
<td>( b^{nx} = (b^x)^n )</td>
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**Proofs.**

\[
\log_b xy = \log_b x + \log_b y \quad \text{iff} \quad b^{\log_b xy} = b^{\log_b x} b^{\log_b y} \quad \text{iff} \quad xy = b^{\log_b x} b^{\log_b y} \quad \text{iff} \quad xy = xy \quad \text{true.}
\]

Likewise for \( \log_b \frac{x}{y} = \log_b x - \log_b y \).

\[
\log_b x^n = n \log_b x \quad \text{iff} \quad b^{\log_b x^n} = b^{n \log_b x} \quad \text{iff} \quad x^n = b^{n \log_b x} \quad \text{iff} \quad x^n = x^n \quad \text{true.}
\]

When \( b = e \), \( \log_{e}(b) = 1 \) becomes: \( \ln(e) = 1 \)

Note: \( \log_b (x \cdot y) = \log_b x + \log_b y \neq \log_b (x + y) \). The last term cannot be broken into simpler pieces.

Simplify.

- \( \ln e^2 - \ln e^4 + \ln 1 - \ln e = 2 - 4 + 0 - 1 = -3 \)
- \( e^{\ln 2} - e^{\ln 4} + e^{\ln 1} - e^{\ln e} = 2 - 4 + 1 - e = -1 - e \)

Combine into a single logarithm.

- \( 2 \log_{10} x + \log_{10} y = \log_{10} x^2 + \log_{10} y = \log_{10} x^2 y \)
- \( \log_{2} x - 4 \log_{2} y = \log_{2} x - \log_{2} y^4 = \log_{2} (\frac{x}{y^4}) \)
- \( \ln(x^2 - y^2) - \ln(y^2 + 1) = \ln \frac{x^2 - y^2}{y^2 + 1} \)

Write as a sum/difference/multiple of the simplest possible logarithms.

- \( \log_b \sqrt{\frac{2y}{y^3-a}} = \log_b \left[ \left( \frac{2y}{y^3-a} \right)^{\frac{1}{2}} \right] = \frac{1}{4} \log_b \frac{2y}{y^3-a} = \frac{1}{4} \left[ \log_b (2y) - \log_b (y^3-a) \right] = \frac{1}{4} \left[ \log_b 2 + \log_b y - \log_b (y^3-a) \right] \)
- \( \ln \sqrt{a^4 + b^4} = \ln(a^4 + b^4)^{-1/2} = -\frac{1}{2} \ln(a^4 + b^4) \)

Solving for \( x \).

- Put terms involving \( x \) on the left, the rest on the right.
- Combine into a single logarithm.
- Exponentiate both sides to the base of the logarithm.
- Solve.
- Delete invalid solutions that give undefined logarithms.

- \( \log_2 4x - \log_2 3 = \log_2 (x + 2) \)
- \( \log_2 4x - \log_2 (x + 2) = \log_2 3 \)
- \( \log_2 \frac{4x}{x+2} = \log_2 3 \)
- \( 2^\frac{\ln x}{x+2} = 2^\log_2 3 \) \( \frac{4x}{x+2} = 3 \)
- \( 4x = 3x + 6 \)
- \( x = 6 \)

Validity: for \( x = 6 \), both \( \log_2 4x \) and \( \log_2 (x + 2) \) are defined, thus the solution is valid.

- \( \ln(x + 1) = 2 - \ln(x - 1) \)
- \( \ln(x + 1) + \ln(x - 1) = 2 \)
- \( \ln [(x + 1)(x - 1)] = 2 \)
- \( \ln(x^2 - 1) = 2 \)
- \( e^{\ln(x^2-1)} = e^2 \)
- \( x^2 - 1 = e^2 \)
- \( x^2 = 1 + e^2 \)
- \( x = \pm \sqrt{1 + e^2} \)

Validity: \( x = -\sqrt{1 + e^2} \) is not valid since \( x \) negative \( \Rightarrow x-1 \) is negative \( \Rightarrow \ln(x-1) \) is undefined.

The only valid solution is \( x = \sqrt{1 + e^2} \).

**CHANGE OF BASE FORMULA.**

\[ \log_a x = \frac{\log_b x}{\log_b a} \]

Proof: \( \log_{b} x = \log_{b} (a^{\log_{a} x}) = \log_{a} (x) \log_{b} a \) now divide both sides by \( \log_{b} a \).

Express in terms of \( \log_{10} \): \( \log_{2} 2 = \frac{\log_{10} 2}{\log_{10} 2} \)

Express in terms of natural logarithms \( \ln \): \( \log_{5} t = \frac{\ln t}{\ln 5} \)