Math 140  Lecture 14

Exponential growth
A bacteria colony starts with 10 bugs. Each bug splits into two bugs every hour. How many bugs are there after $t$ hours?

<table>
<thead>
<tr>
<th>Number of hours</th>
<th>Number of bugs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>$10^2$</td>
</tr>
<tr>
<td>2</td>
<td>$(10^2)\cdot2=10^2\cdot2$</td>
</tr>
<tr>
<td>3</td>
<td>$(10^2)\cdot2=10^2\cdot2^2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$t$ hours</td>
<td>$10^2\cdot2^t$</td>
</tr>
</tbody>
</table>

Exponential functions measure the size of a growing population, the amount of money in a compound interest account, the number of atoms left after radioactive decay, etc. $N(t)$ is the amount at time $t$.

**BASE-e FORM LEMMA.** Every exponential function can be written in the form

$$N(t) = N_0e^{kt}.$$  

- $N_0 = N(0)$ is the initial amount.
- $k$, the coefficient of $t$, is the growth constant.
- If $k > 0$, $N(t)$ measures exponential growth.
- If $k < 0$, $N(t)$ measures exponential decay.

**FACT:** Every $a > 0$ is a power of $e$:

$$a = e^{\ln a}.$$  

Example: $2 = e^{\ln 2}$.

- Write the bug population $N(t) = 10\cdot2^t$ in base-e form.
  $$N(t) = 10\cdot2^t = 10\cdot(e^{\ln 2})^t = 10\cdot e^{[\ln 2]t}$$
  Hence, in base-e form, $N(t) = 10e^{[\ln 2]t}$. Thus $N_0 = 10$, the initial amount and $k = \ln(2)$, the natural log of the initial base.

- You put $4,000$ in an account at 5% interest compounded annually. Write the amount $N(t)$ of money in the account after $t$ years in base-e form.

<table>
<thead>
<tr>
<th>Number of years</th>
<th>Amount $N(t)$ in account</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4,000</td>
</tr>
<tr>
<td>1</td>
<td>$4000\cdot(0.05)\cdot4000 = 4000\cdot(1.05)$</td>
</tr>
<tr>
<td>2</td>
<td>$4000\cdot(1.05)\cdot(1.05)\cdot(1.05) = 4000\cdot(1.05)^3$</td>
</tr>
<tr>
<td>3</td>
<td>$4000\cdot(1.05)^3$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$t$ years</td>
<td>$4000\cdot(1.05)^t$</td>
</tr>
</tbody>
</table>

Hence $N(t) = 4000(1.05)^t$  

$$= 4000(e^{\ln(1.05)})^t = 4000e^{[\ln(1.05)]t}.$$  

- $N(t) = 4000e^{[\ln(1.05)]t}$.
- $N_0 = 4000$ and $k = \ln(1.05)$.

In each problem, first write $N(t)$ in base-e form, then solve.

- A bacteria colony starts with $10^3$ bugs. Four hours later it has $5\times10^3$ bugs.

(a) Find the growth constant $k$.
(b) What is the population two hours after the start?
(c) How long will it take for the population to triple?

First find $N(t) = \text{the number of bugs after } t \text{ hours}.$

**Given:**

<table>
<thead>
<tr>
<th>$t$</th>
<th>$N(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$10^3$</td>
</tr>
<tr>
<td>1</td>
<td>$10^2$</td>
</tr>
<tr>
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</tr>
</tbody>
</table>

**Time:**

- After two hours the population is $4000 = 10\cdot2^2$.

**Solution:**

$$N(t) = N_0e^{kt}.$$  

$10^3e^{5k} = N(4) = 5\cdot10^3$  

$e^{4k} = 5$  

$\ln(e^{4k}) = \ln 5$  

$4k = \ln 5$  

$k = \frac{\ln 5}{4}$

**Then:**

$$N(t) = 10^3e^{\frac{\ln 5}{4}t}.$$  

- **base-e form**

(a) The growth constant is $k = \frac{\ln 5}{4}$.

(b) After two hours the population is

$$N(2) = 10^3e^{\frac{\ln 5}{4}\cdot2} = 10^3e^{\frac{\ln 5}{2}}.$$  

(c) Let $t$ be the time when the population has tripled.

$$N(t) = 3\times\text{the initial amount } N_0.$$  

- $10^3e^{\frac{\ln 5}{4}t} = 3\cdot10^3$

$e^{\frac{\ln 5}{4}t} = 3$

$t = \frac{\ln 5}{\frac{\ln 5}{4}} = \ln 3$

The population triples after $\frac{4\ln 3}{\ln 5}$ hours.

- Initially a sample has 8 lbs of a radioactive substance with a half-life of 5 days, i.e., half decays after 5 days. How many lbs remain after 3 days?  

**First find** $N(t) = \text{the number of lbs left after } t \text{ days}.$

**Given:**

<table>
<thead>
<tr>
<th>$t$</th>
<th>$N(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

**Time:**

- Half (5 days) is enough, so $t = 5$ days.

**Solution:**

$$N(t) = 8e^{kt}.$$  

$8e^{5k} = N(5) = 4$  

$e^{5k} = \frac{1}{2}$  

$5k = \ln \frac{1}{2}$  

$k = \frac{1}{5}\ln(1/2)$

**Then:**

$$N(t) = 8e^{\frac{1}{5}\ln(1/2)t} \quad \text{base e form}$$

$$N(3) = 8e^{\frac{1}{5}\ln(1/2)\cdot3} = 8e^{\frac{3}{5}\ln 1/2}$$

- $8e^{\frac{3}{5}\ln 1/2}$ lbs remain after 3 days.

This is a decay since $\ln(1/2) = \ln(2^{-1}) = -\ln(2)$ is negative.