Math 140  Lecture 18

Exam 3 covers lectures 12 -18. Study the recommended exercises.

**Definition.** For any function \( f \), \( f \) is even iff \( f(-x)=f(x) \).
\( f \) is odd iff \( f(-x)=-f(x) \).

Graphically, the left half (the half plane left of the y-axis) of an even function is of the right half across the y-axis. The left half of an odd function is the negative of this reflection.

\( f(x)=x^2 \) is even since \( f(-x)=(-x)^2=x^2=f(x) \).
\( g(x)=x^3 \) is odd since \( g(-x)=(-x)^3=-x^3=-g(x) \).

![Graph of even and odd functions]

Instead of thinking of \( \sin \theta \) as a function of an angle \( \theta \), we can think of it as a function \( \sin(t) \) of a real variable \( t \).

**Minus Theorem.** \( \sin(-t)=-\sin(t) \), \( \cos(-t)=\cos(t) \), \( \tan(-t)=-\tan(t) \). Thus \( \cos(t) \) is even; \( \sin(t) \) and \( \tan(t) \) are odd functions.

**Proof.**

\[ \tan(-t) = \sin(-t)/\cos(-t) = -\sin(t)/\cos(t) = -\tan(t). \]

**Definition.** A function \( f \) is periodic with period \( p \) iff
\( f(x+p)=f(x) \)
for all \( x \). \( p \) must be the smallest such positive number.

**2π Theorem.** \( \sin(t+2\pi)=\sin(t) \), \( \cos(t+2\pi)=\cos(t) \), for integers \( k \), \( \sin(t+2k\pi)=\sin(t) \), \( \cos(t+2k\pi)=\cos(t) \).

**Proof.** See the picture for \( \sin \) and \( \cos \). \( \tan(t+\pi)=\sin(t+\pi)/\cos(t+\pi)=\sin(t)/(-\cos(t))=\sin(t)/\cos(t) = \tan(t) \).

\[ \cos(t, \sin t) = (\cos(t+2\pi), \sin(t+2\pi)) \]
\[ (-\cos t, -\sin t) = (\cos(t+\pi), \sin(t+\pi)) \]

**π Theorem.** \( \sin(t+\pi)=-\sin(t) \), \( \cos(t+\pi)=-\cos(t) \), \( \tan(t+\pi) = \tan(t) \).

**Proof.**

Thus \( \sin \) and \( \cos \) have period \( 2\pi \), but \( \tan \) has period \( \pi \).

- Simplify: \( \cos(3\pi-x) \).
  \( \cos(3\pi-x)=\cos(2\pi+\pi-x)=\cos(\pi-x)=-\cos(-x)=-\cos(x) \).

Previously we found reference angles in \([0,\pi/2]\).

Rewrite in terms of the nearest multiple of \( \pi \). Throw out this multiple of \( \pi \) using the 2\( \pi \) or \( \pi \) Theorem. Handle minus signs using the Minus Theorem.

**Rewrite with a reference angle in \([0,\pi/2]\), then evaluate.**

\( \sin(5\pi/6) \), \( 5\pi/6=\pi/2+\pi/3 \).
\( \sin(5\pi/6)=\sin(\pi/2+\pi/3)=\sin(\pi/2)\cos(\pi/3)+\sin(\pi/3)\cos(\pi/2)=1\times\cos(\pi/3)+\sin(\pi/3)\times1=\cos(\pi/3)+\sin(\pi/3)=\sqrt{3}/2+\sqrt{3}/2=\sqrt{3}. \)

\( \cos(4\pi/3) \), \( 4\pi/3=\pi+\pi/3 \).
\( \cos(4\pi/3)=\cos(\pi+\pi/3)=-\cos(\pi/3)=-\sqrt{3}/2 \).

\( \tan(-7\pi/4) \).
\( \tan(-7\pi/4)=-\tan(7\pi/4) \).
\( 7\pi/4=\pi+\pi/4 \).
\( \tan(7\pi/4)=-\tan(\pi+\pi/4)=-\tan(\pi+\pi/4)=-\tan(\pi-\pi/4)=-\tan(-\pi/4)=\tan(\pi/4)=\sin(\pi/4)/\cos(\pi/4)=1. \)

\( \sin(21\pi/2) \).
\( \sin(21\pi/2)=\sin(20\pi/2+\pi/2)=\sin(10\pi+\pi/2)=\sin(\pi/2)=1. \)

\( \cos(-3\pi/6) \).
\( \cos(-3\pi/6)=\cos(3\pi/6)=\cos(\pi/2)=0. \)

**Pythagorean Identities.** \( \sin^2 t+\cos^2 t=1 \), \( \tan^2 t+1=\sec^2 t \), \( \cot^2 t+1=\csc^2 t \).

**Proof.** We’ve proved the first. To prove the second, multiply both sides by \( \cos^2 t \). For the third, multiply both sides by \( \sin^2 t \).

- Simplify \( \csc t + \csc t \cot^2 t \)/\( \sec^2 t - \tan^2 t \).
  \( \csc t \left[ 1+\cot^2 t \right]/\left( 1+\tan^2 t \right)-\tan^2 t = \csc t (\csc^2 t)/1 = \csc^3 t \).

- Simplify \( \sin^2 t-\cos^2 t \)/\( \sin^4 t-\cos^4 t \).
  \( \sin^2 t-\cos^2 t \)/\( \left( \sin^2 t-\cos^2 t \right)\left( \sin^2 t+\cos^2 t \right) \) = 1

**Prove cot θ + tan θ = 1/cot θ + tan θ = 1/tan θ.**

Let \( x = \tan θ \), then \( \cot θ = 1/\tan θ = 1/x \).

Thus we must prove
\[ \frac{1}{x} + x + 1 = \frac{1}{x} + x + 1 - \frac{1}{x} \]
\[ \frac{1}{x} + x + 1 = \frac{1}{x} + x + 1 - \frac{1}{x} \]
\[ \frac{1}{x} + x + 1 = \frac{1}{x} - x + 1 \]
\[ \frac{1}{x} + x + 1 = \frac{1}{x} - x + 1 \]
\[ \frac{1}{x} + x + 1 = \frac{1}{x} - x + 1 \]
\[ \text{iff } 1 + x^2 + x = [1-x^3]/(1-x) \]
\[ \text{iff } (x^2+x+1)(1-x) = (1-x^3) \text{ iff } 1-x^3 = 1-x^3 \]
\[ \text{iff true} \]