Math 311 Lecture 6

For today, all matrices will have n rows, for some fixed n. I means I_n.

LEMMA. If B is invertible and $BA = I$, then $A^{-1} = B$.

PROOF. $BA = I \Rightarrow B^{-1}BA = B^{-1}I \Rightarrow A = B^{-1} \Rightarrow$, since being inverse is symmetrical, $A^{-1} = B$. □

Recall the three elementary row operations: swap two rows, multiply a row by a nonzero constant, add a constant multiple of another row.

DEFINITION. If A and B have the same number of rows, $A \vdots B$ consists of the columns of A followed by the columns of B.

LEMMA. For any row operation e, $(eA \vdots eB) = e(A \vdots B)$.

LEMMA. Every elementary row operation is invertible.

PROOF. Swapping two rows twice, gives back the original matrix. The inverse of multiplying a row by a is multiplying it by a^{-1}. The inverse of adding a times another row j is subtracting a times row j. □

DEFINITION. For any row operation e, let $e(A)$ be the result of applying e to A. $E = e(I)$ is an *elementary matrix*.

LEMMA. For any row operation e, the elementary matrix $E = e(I)$ is invertible. For any matrix A, $e(A) = EA$.

LEMMA. A product of elementary matrices is invertible.

PROOF. The product of invertible matrices is invertible. □

- Let e = swap rows 1,2. Then $E = e(I) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$E \cdot \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} r & s \\ p & q \end{pmatrix} = e\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right)$$

Let f = multiply last row by a. $\therefore$ $F = f(I) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$.

$$F \cdot \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} p & q \\ ar & as \end{pmatrix} = f\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right)$$

Let g = add a times first row to last row.

$G = g(I) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$.

$$G \cdot \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} p & q \\ ap+r & aq+s \end{pmatrix} = g\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right)$$

Suppose applying a sequence $e_1, e_2, e_3, \ldots, e_n$ of row operations to A gives B. $\therefore$ $B = e_n(\ldots e_3(e_2(e_1A))\ldots)$. $\therefore$ $B = E_n(\ldots(E_3(E_2(E_1A)))\ldots) = (E_n\ldots E_3E_2E_1)A$. Hence multiplying on left by $(E_n\ldots E_3E_2E_1)$ equals the result of applying the original sequence of operations.

THEOREM. If a sequence of row operations converts $(A \vdots I)$ to $(I \vdots B)$, then $B = A^{-1}$.

PROOF. Suppose $e_n(\ldots e_3(e_2(e_1(A \vdots I))))\ldots) = (I \vdots B)$, $\therefore$ $e_n(\ldots e_3(e_2(e_1(A))))\ldots) = I$ & $e_n(\ldots e_3(e_2(e_1(I)))\ldots) = B$. $\therefore$ $(E_n\ldots E_3E_2E_1)A = I$ and $(E_n\ldots E_3E_2E_1)I = B$. By the lemma above, $A^{-1} = E_n\ldots E_3E_2E_1 = E_n\ldots E_3E_2E_1I = B$. □

- Find the inverse of $A = \begin{bmatrix} 0 & 0 & 4 \\ 2 & 4 & 0 \\ 3 & 3 & 0 \end{bmatrix}$.

$$(A \vdots I) = \begin{bmatrix} 0 & 0 & 4 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ 3 & 3 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{pmatrix} \mathbf{1} \\ <> \\ \mathbf{3} \end{pmatrix}\begin{bmatrix} 3 & 3 & 0 & 0 & 0 & 1 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{pmatrix} \mathbf{1}{*}{=}\ \frac{1}{3} \\ \mathbf{2}{*}{=}\ \frac{1}{2} \\ \mathbf{3}{*}{=}\ \frac{1}{4} \end{pmatrix}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \frac{1}{3} \\ 1 & 2 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & 0 & 0 \end{bmatrix} \rightarrow \begin{pmatrix} \mathbf{2} \\ -= \\ \mathbf{1} \end{pmatrix}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{-1}{3} \\ 0 & 0 & 1 & \frac{1}{4} & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{pmatrix} \mathbf{1} \\ -= \\ \mathbf{2} \end{pmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{-1}{2} & \frac{2}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{-1}{3} \\ 0 & 0 & 1 & \frac{1}{4} & 0 & 0 \end{bmatrix}. \quad \text{Hence } A^{-1} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{2}{3} \\ 0 & \frac{1}{2} & \frac{-1}{3} \\ \frac{1}{4} & 0 & 0 \end{bmatrix}.$$

THEOREM. If A reduces to I, then A is a product of elementary matrices.

PROOF. Reduce $(A \vdots I)$ to $(I \vdots B)$ by a sequence $e_1, e_2, \ldots, e_n$ of elementary row operations. Hence $A^{-1} = B = E_n\ldots E_2E_1$. Hence $A = B^{-1} = (E_n\ldots E_2E_1)^{-1} = E_1^{-1}E_2^{-1}\ldots E_n^{-1} = D_1D_2\ldots D_n$ where D_i is the elementary matrix for the inverse of the row operation e_i. □

- Write A (see example above) as an elementary product.

$$A = \begin{pmatrix} \mathbf{1} \\ <> \\ \mathbf{3} \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{1} \\ {*}{=} \\ \frac{1}{3} \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{2} \\ {*}{=} \\ \frac{1}{2} \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{3} \\ {*}{=} \\ \frac{1}{4} \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{2} \\ -= \\ \mathbf{1} \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{1} \\ -= \\ \mathbf{2} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hw 4 Answers

16. (a) $A+A^T$ is symmetric: $(A+A^T)^T = A^T + A^{TT} = A^T + A = A + A^T$.
(b) $A-A^T$ skew symmetric: $(A-A^T)^T = A^T - A^{TT} = A^T - A = -(A-A^T)$.

18. Suppose A and B are symmetric.
(a) Show that A+B is symmetric: $(A+B)^T = A^T + B^T = A + B$.
(b) Show that AB is symmetric iff AB=BA:
AB symmetric iff $(AB)^T = AB$ iff $B^TA^T = AB$ iff $BA = AB$.

22. $D^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{-1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$. 24. $A = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 2 & -1 \end{bmatrix}$.

34. Suppose AB = AC and A is nonsingular. Prove B = C.
$AB = AC \Rightarrow A^{-1}AB = A^{-1}AC \Rightarrow IB = IC \Rightarrow B = C$.

36. Suppose A is symmetric and nonsingular. Prove A^{-1} symmetric.
$(A^{-1})^T$ =,(theorem, previous lecture), $(A^T)^{-1}$ =,(A symmetric), A^{-1}.

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$$X = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{pmatrix} Z = XY.$$

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}, Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}.$$

$$Z_1 = X_1Y_1 + X_2Y_3 = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \qquad Z_2 = X_1Y_2 + X_2Y_4 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$Z_3 = X_3Y_1 + X_4Y_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad Z_4 = X_3Y_2 + X_4Y_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$Z = \begin{pmatrix} 4 & 4 & 2 & -1 \\ 4 & 4 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$