The critical value \( z_\alpha \) is defined by \( P(z > z_\alpha) = \alpha \). Look up \( 1-\alpha \) inside the normal table to get \( z_\alpha \) (\( P(z < z_\alpha) = 1 - \alpha \)) or find \( \alpha \) inside the table and get \(-z_\alpha\) (\( P(z < -z_\alpha) = \alpha \)).

**Definition.** For a normal sample average \( \bar{x} \) and margin of error \( b, \bar{x} \pm b = [\bar{x} - b, \bar{x} + b] \) is the 95% confidence interval. For any confidence level \( \beta \) (e.g., \( \beta = 90\% \)), the \( \beta \) confidence interval is the interval centered around \( \bar{x} \) which contains the true average \( \mu \) with probability \( \beta \).

\( \alpha = 1 - \beta \) is the significance level. Thus the confidence interval endpoints are \( z_{a/2} \) std. devs. above and below the mean.

Convention: write \( \alpha \) as a decimal and \( \beta \) as a percentage. If \( \alpha = .01 \), then \( \beta = (1-\alpha) = .99 = 99\% \).

**Theorem.** For a confidence level \( \beta \) and significance level \( \alpha = 1 - \beta \), the \( \beta \) confidence interval for the actual population mean \( \mu \) is

\[
\mu \in \bar{x} \pm z_{a/2} \times SE = [\bar{x} - z_{a/2} \times SE, \bar{x} + z_{a/2} \times SE].
\]

The 95% confidence interval for \( \mu \) is

\[
\mu \in \bar{x} \pm z_{.025} \times SE = [\bar{x} - 1.96 \times SE, \bar{x} + 1.96 \times SE].
\]

The true mean \( \mu \) is in this interval with probability 95%.

5/2% of the population is above the 95% confidence interval and 5/2% is below. Looking up 5/2% = .025 inside the normal table gives -1.96. Thus for \( \alpha = .05 \), \( z_{a/2} = z_{.025} = 1.96 = \) the margin-of-error.

If \( \beta = 99\% \), \( \alpha = (1-\beta) = .01 \), \( \alpha/2 = .005 \). Locate .005 inside the normal table to get -2.58. \( z_{a/2} = 2.58 \). 99% of the time, the correct answer \( \mu \) is in \( \bar{x} \pm 2.58 \times SE \).

Memorize the critical values, 1.645, 1.96 and 2.58, of \( z_{a/2} \) for the 90%, 95% and 99% confidence intervals.

- In a sample of \( n = 100 \) measurements, the mean \( \bar{x} = 40 \) and the sample std. dev. is \( s = 3 \). Find the 99% confidence interval for \( \mu \) around \( \bar{x} \).

The SE for the estimator \( \bar{x} \) is \( \frac{s}{\sqrt{n}} = \frac{3}{10} = .3 \) For 99% confidence, \( z_{a/2} = 2.58 \). Thus the interval is

\[
\bar{x} \pm 2.58 \times SE = \bar{x} \pm 2.58 \times SE
\]

Answer: \( \mu \in [40 \pm (2.58) \times .3] = [39.23, 40.77] \).

- In a sample of \( n = 100 \) measurements, the proportion of successes is \( \hat{p} = .40 \). Find the 99% confidence interval for \( p \) around the sample estimate of .40. \( p \in \hat{p} \pm 2.58 \times SE = .4 \pm 2.58 \sqrt{\frac{\hat{p}(1-\hat{p})}{100}} = .4 \pm 2.58 \times .049 = [.27, .53] \).

**Estimating Differences of Means**

To estimate the difference between UH males and females w.r.t. SAT scores, we first pick a random sample of males and calculate their average SAT score \( \bar{x}_1 \) and then pick a sample of females and calculate their average SAT score \( \bar{x}_2 \). Then \( \bar{x}_1 - \bar{x}_2 \) is the obvious estimate of the difference between male and female SAT scores.

**Theorem.** Given two populations with means \( \mu_1, \mu_2 \) and std. devs. \( \sigma_1, \sigma_2 \) and given two samples taken from the respective populations with \( n_1, n_2 \) elements and with sample means \( \bar{x}_1, \bar{x}_2 \) respectively: the difference \( \bar{x}_1 - \bar{x}_2 \) is an unbiased estimator of \( \mu_1 - \mu_2 \), \( E(\bar{x}_1 - \bar{x}_2) = \mu_1 - \mu_2 \).

The std. dev. of \( \bar{x}_1 - \bar{x}_2 \) is \( SE = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \).

**Proof of SE.** The sample std. dev. for the men is \( \frac{\sigma_1}{\sqrt{n_1}} \) with variance \( \frac{\sigma_1^2}{n_1} \). The women’s std. dev. is \( \frac{\sigma_2}{\sqrt{n_2}} \). Using the formulas \( \text{Var}(x_1 + x_2) = \text{Var}(x_1) + \text{Var}(x_2) \) and \( \text{Var}(ax) = a^2 \text{Var}(x) \), we get:

\[
\text{Var}(x_1 - x_2) = \text{Var}(x_1 + (-1)x_2) = \text{Var}(x_1) + \text{Var}((-1)x_2) = \text{Var}(x_1) + (-1)^2 \text{Var}(x_2) = \text{Var}(x_1) + \text{Var}(x_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}.
\]

Hence the std. dev. of \( x_1 - x_2 \) is \( SE = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \). \( \square \)

For approximately normal populations or for \( n_1, n_2 \geq 30 \), \( \bar{x}_1 - \bar{x}_2 \) is also approximately normal (can use normal tables).

Statistics can never prove that two populations are equal. One can only determine that there is a significant difference or that there is no significant difference at a given confidence level.

The 95% confidence interval around \( \bar{x}_1 - \bar{x}_2 \) contains the difference \( \mu_1 - \mu_2 \) of the true means with probability 95%.

\( 0 \notin \) the 95% confidence interval for \( \mu_1 - \mu_2 \)

\( \Rightarrow \mu_1 - \mu_2 \) could be 0

\( \Rightarrow \mu_1 \) and \( \mu_2 \) do not differ significantly.

\( 0 \notin \) the 95% confidence interval for \( \mu_1 - \mu_2 \)

\( \Rightarrow P(\mu_1 - \mu_2 < 5) < 5\% \)

\( \Rightarrow P(\mu_1 - \mu_2 > 5) < 5\% \Rightarrow \mu_1 \) and \( \mu_2 \) do differ significantly.

Warning, if 0 is in the 95% confidence interval, it could be (1) because means are equal or (2) because the sample sizes are not large enough to detect the difference.

- Males and females are tested for verbal ability.

<table>
<thead>
<tr>
<th>Sample size</th>
<th>Males</th>
<th>Females</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>100</td>
<td>105</td>
</tr>
<tr>
<td>40</td>
<td>20</td>
<td>10</td>
</tr>
</tbody>
</table>

Are females significantly better at 95% confidence level?

\( \bar{x}_1 - \bar{x}_2 = -5, SE = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{20^2}{30} + \frac{10^2}{40}} = 3.98 \).

95% confid. int.: \( \mu_1 - \mu_2 = -5 \pm 1.96 \times 3.98 = [-12.80, 2.80] \).

No significant difference. Why? \( \mu_1 - \mu_2 = 0 \in [-12.80, 2.80] = 95\% \) confidence interval. The difference isn’t large enough to be significantly different from 0 at this level of confidence.