Math 373 Lecture 23

Pooled sample deviations

Recall: We have two samples with n_1, n_2 elements, with means x_1, x_2 and sample std. devs. s_1, s_2. The usual estimate for the std. dev. of $\bar{x}_1-\bar{x}_2$ is: SE $=\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}$. Since a null hypothesis that $\mu_1=\mu_2$ usually also implies that $\sigma_1=\sigma_2$, we replace the two sample std. devs. s_1 and s_2 with a more accurate pooled estimate of σ:

$$s=\sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}}.$$

Using s in place of s_1 and s_2 gave an improved estimate of SE, the standard deviation of $\bar{x}_1-\bar{x}_2$:

$$\text{SE}=\sqrt{\frac{s^2}{n_1}+\frac{s^2}{n_2}}=\sqrt{s^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}=s\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}.$$

When n_1+n_2-2 is small (<30) use the t distribution with $df=n_1+n_2-2$ which is the denominator of s.

■ Samples are taken from two normal populations to test for a 1% significant difference between their means.

	Pop 1	Pop 2
Sample size	10	15
Sample mean	50	45
Sample std. dev.	2	3, *4

The null and alternate hypotheses.

H_a: $\mu_1\neq\mu_2$, $\mu_1-\mu_2\neq 0$, $t\neq 0$. H_0 : $\mu_1=\mu_2$, $\mu_1-\mu_2=0$, $t=0$.

The pooled std. dev. s and the standard error.

$$\bar{x}_1-\bar{x}_2=5,\quad s=\sqrt{\frac{(9)2^2+(14)3^2}{10+15-2}}=2.6540$$

$$\text{SE}=2.654\sqrt{\frac{1}{10}+\frac{1}{15}}=1.0835$$

$$t=\frac{(\bar{x}_1-\bar{x}_2)-0}{SE}=4.615,\quad df=23,\quad t_{\alpha/2}=2.807,$$

1% acceptance region for $\bar{x}_1-\bar{x}_2$: $\bar{x}_1-\bar{x}_2\in$

$\mu_1-\mu_2\pm t_{a/2}\text{SE}=(0)\pm(2.807)(1.083)=[-3.0413, 3.0413]$.

1% acceptance region for the t value of $\bar{x}_1-\bar{x}_2$:

$t\in 0\pm t_{a/2}=0\pm(2.807)=[-2.807, 2.807]$.

Conclusion: Reject H_0. The difference is significant.

Exception: Suppose the null hypothesis is H_0: $\mu_1=\mu_2$, but the larger of the two variances s_1^2, s_2^2 is more than three times the size of the smaller. Then abandon the assumption that $\sigma_1=\sigma_2$ and use the original unpooled formula SE $=\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}$.

In the pooled case, we used n_1+n_2-2 as the degrees of freedom df in the table for Student's t distribution. But if we abandon the assumption that $\sigma_1=\sigma_2$, the elements in the sample with the larger std. dev. vary more, they contribute more to the error and hence they are responsible for a larger proportion of the total number of degrees of freedom. Here is the awful formula which appropriately weights the first sample's n_1-1 degrees of freedom with the second's n_2-1 degrees. Round the resulting df to the nearest integer.

$$df=\frac{(v_1+v_2)^2}{\frac{v_1^2}{n_1-1}+\frac{v_2^2}{n_2-1}}\quad\text{where } v_1=\frac{s_1^2}{n_1},\ v_2=\frac{s_2^2}{n_2}.$$

■ In the previous example, suppose the second population has sample std. dev. $s_2=4$ rather than 3.

Then $s_2^2/s_1^2=16/4=4>3$. Hence use the unpooled SE.

$$\text{SE}=\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}=\sqrt{\frac{2^2}{10}+\frac{4^2}{15}}=1.2111$$

$$df: v_1=\frac{2^2}{10},\ v_2=\frac{4^2}{15},\ df=\frac{(v_1+v_2)^2}{v_1^2/(n_1-1)+v_2^2/(n_2-1)}=21.7\to 22.$$

$\bar{x}_1-\bar{x}_2=5$, $t_{a/2}=2.819$.

Acceptance region for $\bar{x}_1-\bar{x}_2$.

$\bar{x}_1-\bar{x}_2\in[0\pm 2.819\times 1.2111]=[-3.41, 3.41]$.

Conclusion: Reject H_0. The difference is significant.

Why? $\bar{x}_1-\bar{x}_2=5\notin[-3.41, 3.41]=$ acceptance.

Paired difference tests

■ To test the durability of tires of brand A compared with tires of brand B, you put tires A and B on the rear wheels of 4 cars, two cars have A on the right, B on the left, the other two have B on the right, A on the left. After 20,000 miles you measure the amount of wear in millimeters.

	Brand A	Brand B	difference
car 1	6.2	6.0	.2
car 2	3.1	3.0	.1
car 3	4.6	4.5	.1
car 4	4.7	4.5	.2
average	$\bar{x}_1=4.65$	$\bar{x}_2=4.5$	$d=.15$
std. dev.	$s_1=1.27$	$s_2=1.22$	$s_d=0.06$

LEMMA. The difference x_1-x_2 of the sample means equals the mean x_d of the sample differences.

$$\frac{(a_1-b_1)+(a_2-b_2)+(a_3-b_3)}{3}=$$
$$\frac{(a_1+a_2+a_3)-(b_1+b_2+b_3)}{3}=\frac{a_1+a_2+a_3}{3}-\frac{b_1+b_2+b_3}{3}.$$

Hence the null hypothesis $\bar{x}_1-\bar{x}_2=0$ is equivalent to the null hypothesis $x_d=0$ where d is the average difference.

However the two std. devs. for the two brands is much worse than the std. dev. for the differences. You get much more accurate results using paired differences.

Paired differences give less error because driving habits affect tire wear more than choice of tire brand. When tires of different brands are on cars with different drivers, the difference in wear is mostly due to the drivers. When two tires are put on the same car, the difference in wear will be due almost entirely to the tire brand, not the driver.