Math 414 Lecture 3

CORRECTION(MICHAEL). In SciLab, eye(n) is written eye(n,n).

THEOREM. If $v_1,\ldots,v_k$ are k vectors in an n-dimensional space and $A=[v_1; \ldots; v_k]$:

$v_1, \ldots, v_k$ independent $\Rightarrow k \le n$

$v_1, \ldots, v_k$ span the space $\Rightarrow n \le k$

$v_1, \ldots, v_k$ a basis $\Rightarrow k = n$

$v_1, \ldots, v_k$ independent $\Leftrightarrow \text{rank}(A) = k$

$v_1, \ldots, v_k$ span the space $\Leftrightarrow \text{rank}(A) = n$

$v_1, \ldots, v_k$ is a basis $\Leftrightarrow k = \text{rank}(A) = n$

- Classify the set $u=[1,2,3]$, $v=[4,5,6]$, $w=[7,8,9]$ as (a) spanning/nonspanning and (b) dependent/independent.

Solution. $A=[1,2,3; 4,5,6; 7,8,9]$,
rref$(A)=[1,0,-1; 0,1,2; 0,0,0]$, rank$(A)$=2.
$\therefore$ u, v, w are dependent and don't span the space R^3.

- Write $\boldsymbol{b}=[1,1,1]$ as a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ if possible.

First transpose $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$, and $\boldsymbol{b}$. Thus $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w},\boldsymbol{b}$ become column vectors. Let $\boldsymbol{A}=[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]$. $\boldsymbol{b}$ is a linear combination iff $x\boldsymbol{u}+y\boldsymbol{v}+z\boldsymbol{s}=\boldsymbol{b}$ for some x, y, z iff $\boldsymbol{A}$*$[x; y; z]=\boldsymbol{b}$ has a solution.

$[\boldsymbol{A}\,|\,\boldsymbol{b}]$= rref($[\boldsymbol{A}\,|\,\boldsymbol{b}]$)=

x	y	z			x	y	z	
1	4	7	1		1	0	-1	-⅓
2	5	8	1		0	1	2	⅓
3	6	9	1		0	0	0	0

Basic solution: $x=-⅓$, $y=⅓$, $z=0$.
Thus [1 1 1] = −⅓[1 2 3]+⅓[4 5 6]. Check this.

- Write $\boldsymbol{b}=[1\ 0\ 1]$ as a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$.

$[\boldsymbol{A}\,|\,\boldsymbol{b}]$= rref($[\boldsymbol{A}\,|\,\boldsymbol{b}]$) =

x	y	z			x	y	z	
1	4	7	1		1	0	-1	0
2	5	8	0		0	1	2	0
3	6	9	1		0	0	0	1

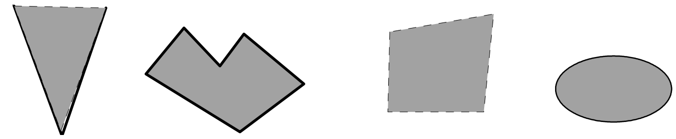
There is no solution (the last line is $0=1$).
Thus [1 0 1] is not a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$.

- Given points $x,y \in R^n$ where is $½x+½y$? $⅓x+⅔y$?

LEMMA. The line segment $\overline{xy}$ between x and y = the set of points between x and y = set of points of the form $ax+by$ where $a+b=1$ and $a, b \ge 0$.

DEFINITION. For any subset S of R^n:
S is *convex* iff $x,y \in S \Rightarrow$ all points between x & y are in S. S is *closed* iff it contains all points on its boundary.
S is *unbounded* iff it has points arbitrarily far apart.
S is *bounded* otherwise.
An *extreme* point (or *vertex*) of S is a point of S which is not between two other points of S.

- Mark the following as conveX, cLosed, or Bounded. Circle the extremes.

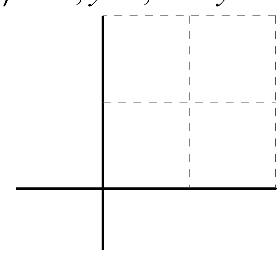


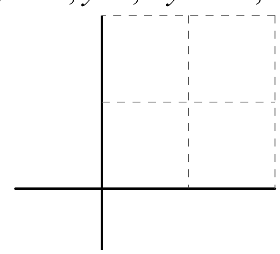
LEMMA. The intersection of convex sets is convex.
The set of solutions to a linear equation is a line in R^2, a plane in R^3, a *hyperplane* in R^n for $n \ge 3$.

DEFINITION. A *closed half-space* is the set of solutions to a non-strict (i.e., ≤ or ≥) linear inequality. E.g. $y \le x+1$.
A *convex polyhedron* is an intersection of finitely many closed half-spaces.
Any half-space is convex, hence so is any intersection of half-spaces.

- Graph the convex polyhedra. First graph the equalities. Then fill in the polyhedra. Circle & label the extremes.

(a) $x \ge 0$, $y \ge 0$, 1: $x+y \le 1$ (b) $x \ge 0$, $y \ge 0$, 1: $y-x \le 1$, 2: $y+x \ge 1$





An inequality determines a half-plane. The half plane lies on the origin's side of the line iff x = 0 and y = 0 satisfy the inequality.

A *convex combination* of points $x_1,\ldots,x_n$ is a point of the form $a_1x_1+a_2x_2+\ldots+a_n x_n$ where $a_1+\ldots+a_n=1$ and $a_1 \ge 0, \ldots, a_n \ge 0$.

The *convex hull* of a given set of points is the set of all convex combinations of points from the given set. The convex hull can also be defined as the smallest convex set containing the given set.

A *convex polytope* is the convex hull of a finite set.

- $\overline{xy}$ = the convex hull of x and y
  = the line segment between x and y.

THEOREM. A closed bounded convex set is the convex hull of its extreme points. For closed bounded convex sets S, S is a convex polytope iff S has only finitely many extreme points iff S is a convex polyhedron.

THEOREM. If f is a linear function on a closed bounded convex set, then every local minimum (or maximum) is also an absolute minimum (or maximum). If f is not constant, the maxima (minima) lie on the boundary.

In a polytope, extremes are not convex combinations of other points but every point is a convex combination of extremes. Thus extremes form a type of a basis.