Math 414  Lecture 4

**Theorem.** Every general linear programming problem can be rewritten in standard and in canonical form.

**Proof.**

- **Rewriting min in terms of max.**
  - $x$ minimizes $f(x)$ iff $-x$ maximizes $-f(x)$

- **Rewriting $\geq$ with $\leq.$**
  - $ax \geq b$ iff $-ax \leq -b$

- **Rewriting $=$ with $\leq.$**
  - $ax = c$ iff $ax \leq c$ and $ax \geq c$

- **Rewriting $\leq$ with $\geq$ and $s \geq 0.$**
  - $ax \leq b$ has a solution iff $ax + s = b$, $s \geq 0$ has a solution (let $s = |b - ax|$).

  $s = |b - ax|$ is a slack variable - it is the distance or slack between $ax$ and $b$.

- **Rewriting $\geq$ with $\leq$ and $s \geq 0.$**
  - $ax \geq b$ has a solution iff $ax - s = b$, $s \geq 0$ has a solution (let $s = |b - ax|$).

- **Reducing unrestricted variables to nonnegatives.**
  - $ax \leq b$ with $x$ unrestricted has a solution iff $a(u - v) \leq b$ and $u \geq 0$, $v \geq 0$ has a solution.
  
  Every number $x$ is the difference $u - v$ of two nonnegatives. $3 = (3) - (0)$, $-3 = (0) - (3)$.

- **Convert to standard and to canonical form.**
  - $\min \{x - 2y \}$
  - $\max -z = \max -z = \max -z =
  
  with
  
  $x + y = 8$

  $x - y \geq 4$

  $y \geq 0$

  $x$ arbitrary

  By multiplying by -1 if necessary, we can change any canonical form into one in which the constant on the right is always $> 0$. Replace $-x + y = -3$ with $x - y = 3$.

  **Lemma.** There is a 1-1 onto correspondence between the feasible solutions of a standard problem and feasible solutions of the canonical version. Every standard point $S$ lifts up to a canonical point by adding the slack variables. Every canonical point projects down to a standard point by deleting the slack variables. E.g., $[x, y] \in$ standard problem $[x, y, s] \in$ canonical. The objective function has no slack variables; it is the same in both cases.

- **Standard problem**
  - $\max w = x + 2y$
  
  with
  
  $s: x + y \leq 3$

  $x, y \geq 0$

  A slack $s$ equals 0

  iff $[x, y]$ is on the boundary of constraint $s$.

  A standard variable $x$ equals 0

  iff the solution $(x, y)$ is on the boundary of its $x \geq 0$

  “constraint”.

  **Theorem.** A canonical variable is zero iff the standard point is on the boundary for that variable.

- **For a 2-variable standard problem,**
  - an edge point has 1 canonical 0 coordinate;
  - an extreme vertex point has 2 canonical 0’s
  - an interior point has no 0 coordinates.

**Corollary.** The number of 0’s in a canonical solution = the number of boundaries on which it lies.

- Interior points have no zero coordinates.

- A standard point is extreme iff it is on more boundaries than nearby points iff it has more canonical zero coordinates than nearby points.

- In the canonical problem, extremes have the more zeros than nearby points.
\[(x,y) = \text{standard point} \quad (x,y,s) = \text{canonical point}.\]

**Standard table**

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**Canonical table**

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Except for the slack variable \(s\), the tables are the same.

In the problem above, either standard or canonical, there is 1 constraint (we don’t count the \(x, y \geq 0\) conditions as constraints) and each extreme value has at most one positive entry. The rest are 0. In general we have

Recall that for a system of \(m\) equations in rref form, there are \(m\) basic variables which might be nonzero but the other variables are parameters which are 0 in the basic solution. Hence basic solutions have at most \(m\) nonzeros.

**Theorem.** In a standard or canonical problem with \(m\) constraints, each extreme feasible solution has at most \(m\) nonzero (and hence positive) entries. The rest are 0.

**Examples.**

- Feasible canonical basic solutions are the extremes.
  - Original constraint: \(s : x + y \geq -2, \ x, y \geq 0\). (Note: redundant.)
  - Canonical version: \(s : x + y - s = -2, \ x, y, s \geq 0\)

  If \(x\) is the basic variable, then \(y, s\) are parameters and the basic solution is: \(x = -2, \ y = 0, \ s = 0\) which is not feasible.

  If \(s\) is the basic variable, then, setting the parameters \(x, y\) to zero gives: \(x = 0, \ y = 0, \ s = 2\) which is feasible.