THEOREM. Every general linear programming problem can be rewritten in standard and in canonical form.

PROOF.

- Rewriting min in terms of max.
  \( x \) minimizes \( f(x) \) iff \( x \) maximizes \( -f(x) \)

- Rewriting \( \geq \) with \( \leq \).
  \( ax \geq b \) iff 
  \(-ax \leq -b \)

- Rewriting \( = \) with \( \leq \).
  \( ax = c \) iff 
  \( ax \leq c \) & \( ax \geq c \) iff 
  \( ax \leq c \) & \(-ax \leq -c \)
- Rewriting $\leq$ with $=$ and a slack variable $s \geq 0$.
  
  $ax \leq b$ has a solution iff
  
  $ax + s = b$, $s \geq 0$ has a solution (let $s = |b - ax|$).

  $s = |b - ax|$ is a *slack variable* - it is the distance or slack between $ax$ and $b$.

- Rewriting $\geq$ with $=$ and a slack variable $s \geq 0$.
  
  $ax \geq b$ has a solution iff
  
  $ax - s = b$, $s \geq 0$ does. (let $s = |b - ax|$).

- Reducing unrestricted variables to nonnegatives.
  
  $ax \leq b$ with $x$ unrestricted has a solution iff
  
  $a(u - v) \leq b$ & $u \geq 0$, $v \geq 0$ has a solution.

  Every number $x$ is the difference $u - v$ of two nonnegatives.
  
  $3 = (?) - (?)$, $-3 = (?) - (?)$.

- Rewrite using nonnegative variables:
  
  $2x \leq 5$

  $x$ unrestricted
**Standard ≤ form:**

maximize

\[ z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \]

subject to

\[ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \]
\[ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \]

\[ \vdots \]
\[ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \]

\[ x_1 \geq 0, \ x_2 \geq 0, \ldots, \ x_n \geq 0. \]

**Canonical = form:**

maximize

\[ z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \]

subject to

\[ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \]
\[ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \]

\[ \vdots \]
\[ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \]

\[ x_1, \ x_2, \ldots, \ x_n \geq 0. \]

**Abbreviation.** \( x, y, u, v \geq 0 \) means \( x \geq 0, \ y \geq 0, \ u \geq 0, \ v \geq 0. \)

- Convert to **Standard form**

\[ \begin{align*}
\min \ z &= x - 2y \\
\text{with} \quad x + y &= 8 \\
x - y &\geq 4 \\
y &\geq 0 \\
x &\text{ arbitrary}
\end{align*} \]
Convert to Canonical form.

\[
\begin{align*}
\text{min } z &= x - 2y \\
\text{with} \\
&\quad x + y = 8 \\
&\quad x - y \geq 4 \\
&\quad x, y \geq 0
\end{align*}
\]

Canonical problems can always be rewritten with nonnegative constants on the right.

- **Rewrite with positive constant on right:**
  
  \(-x + y = -3\)
**Lemma.** There is a 1-1 onto correspondence between the feasible solutions of a standard problem and feasible solutions of the canonical version.

Every standard point \((x, y)\) *lifts up* to a canonical point \((x, y, s)\) by adding the slack variables.

Every canonical point \((x, y, s)\) *projects down* to a standard point \((x, y)\) by deleting the slack variables.

\((x, y) \in \text{standard problem} \iff (x, y, s) \in \text{canonical}\).

The objective function has no slack variables; it is the same in both cases.

<table>
<thead>
<tr>
<th><strong>Standard problem</strong></th>
<th><strong>Canonical version</strong></th>
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</thead>
<tbody>
<tr>
<td>(\max w = x + 2y)</td>
<td>(\max \ ???)</td>
</tr>
<tr>
<td>with (s: x + y \leq 3)</td>
<td>with (???)\</td>
</tr>
<tr>
<td>(x, y \geq 0)</td>
<td>(???)\</td>
</tr>
</tbody>
</table>
\[ s: \ 2x + y \leq 3 \]
\[ t: \ x + 2y \leq 3 \]
\[ x, y \geq 0 \]

*Standard variables:* \( x, y \).  
*Slacks:* \( s, t \).

*Canonical variables:* \( x, y, s, t \).

A slack \( s \) equals 0  
iff the solution \((x, y)\) is on the boundary of constraint \( s \).

A standard variable \( x \) equals 0  
iff the solution \((x, y)\) is on the boundary of its \( x \geq 0 \) “constraint”.

**Theorem.** A canonical variable is zero iff the standard point is on the boundary for that variable.

- For a 2-variable standard problem,  
an edge point has 1 canonical 0 coordinate;  
an extreme vertex point has 2 canonical 0’s  
an interior point has no 0 coordinates.

**Corollary.** The number of 0’s in a canonical solution = the number of boundaries on which it lies.
- Interior points have (no? all?) zero canonical variables.

- A standard point is extreme iff it is on (more? fewer?) boundaries than nearby points iff it has (more? fewer?) canonical zero coordinates than nearby points.

- In canonical problems, extremes have (more? fewer?) zeros than nearby points.
• Interior points have no zero coordinates.

• A standard point is extreme iff it is on more boundaries than nearby points iff it has more canonical zero coordinates than nearby points.

• In the canonical problem, extremes have the more zeros than nearby points.
Find the coordinates of the standard extremes and of the canonical extremes.

\[ s: x + y \leq 3 \quad x, y \geq 0 \]
There is one constraint \((x, y \geq 0)\) are not counted as constraints). Each extreme has exactly \((0?, 1?, 2?, 3?)\) positive canonical entries.
For a system of $m$ independent equations in rref form, how many basic variables which might be nonzero? The other variables are parameters (they are 0 in basic solutions). How many (at most) nonzeros can basic solution have?

**Theorem.** In a problem with $m$ constraints, each extreme feasible solution has at most $m$ nonzero entries. The rest are 0.

**Examples.**

<table>
<thead>
<tr>
<th>No Constraint</th>
<th>$x + y \leq 3$</th>
<th>$x/2 + y \leq 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0,0)$</td>
<td>$(0,3)$</td>
<td>$(0,3)$</td>
</tr>
<tr>
<td>$x, y \geq 0$</td>
<td>$x, y \geq 0$</td>
<td>$x, y \geq 0$</td>
</tr>
</tbody>
</table>
s: \[2x + y \leq 6\]
\[t: \ x + 2y \leq 6\]
\[x, y \geq 0\]
Standard variables: \(x, y\).
Slacks: \(s, t\).
Canonical variables: \(x, y, s, t\).

Label each edge with its canonical variable.
\[ s: 2x + y \leq 6 \]
\[ t: x + 2y \leq 6 \]
\[ x, y \geq 0 \]

Standard variables: \( x, y \).
Slacks: \( s, t \).
Canonical variables: \( x, y, s, t \).

Classify \textit{w.r.t.} the number of canonical variables that are 0:
- Interior points have (\_\_\_\_) 0’s,
- Edge points have (\_\_\_\_) 0’s,
- Extremes have (\_\_\_\_) 0’s.
Feasible canonical basic solutions are the extremes.

Original constraint: \( s : x + y \geq -2, \)
\( x, y \geq 0. \) (Note: redundant.)

Canonical version: \( s : x + y - s = -2, \)
\( x, y, s \geq 0 \)

If \( x \) is the basic variable, the basic solution is \( (?, ?, ?) \)
Is it feasible? \( \text{(yes, no)} \)

Pick a variable to be basic whose basic solution is feasible.
\( (x?, y?, s?) \) \( \square \)