Math 414 Lecture 5

- Maximize $w = x+y+z$ subject to:

$$\begin{array}{rrrrl} 1: & 2x & +2y & +z & \le 6 \\ 2: & x & +y & +2z & \le 6 \\ & x, & y, & z & \ge 0 \end{array}$$

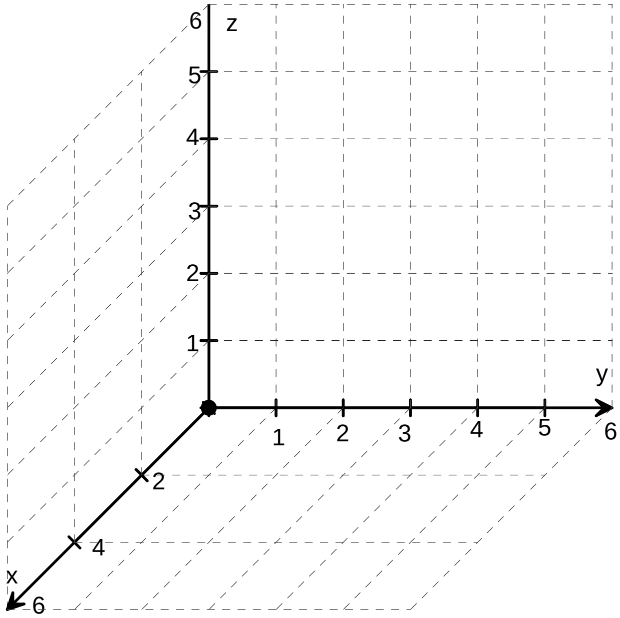
To graph the boundary plane of the <u>equation</u> for 1: $2x+2y+z=6$, find the x, y and z-axis intercepts.

x-axis intercept: set $y=z=0$ and solve for x. Get $x=3$. y-axis intercept: $y=3$, z-axis intercept: $z=6$.

Likewise graph the plane for constraint 2:. These planes intersect in a line. Find this line by simultaneously solving $2x+2y+z=6$ and $x+y+2z=6$. Take the rref of the augmented matrix for this system.

rref(augmented matrix) = Planar intercepts:(2,0,2), (0,2,2).

$$\begin{array}{cccc} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array} \qquad \begin{array}{l} x+y = 2 \\ z = 2 \end{array}$$



x	y	z	w

<u>*General*</u> *linear programming problems*: Find $x_1, x_2, \ldots, x_n$ which maximizes or minimizes a given linear *objective function* subject to a finite set of linear *constraints*.

maximize (minimize) $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ ←objective

with $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le (\ge)(=)\ b_1$ ← constraints

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le (\ge)(=)\ b_2$

...

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le (\ge)(=)\ b_m$.

A *feasible solution* is a point satisfing the constraints. The feasible solutions form a convex polyhedron.

An *optimal solution* is one which maximizes (minimizes) the objective function.

STANDARD AND CANONICAL FORMS

<u>*Standard*</u> ($\le$ *form*) problems:	<u>*Canonical*</u> (=*form*) problems:
maximize	maximize
$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$	$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$
subject to	subject to
$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1$	$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2$	$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$
...	...
$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m$	$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$
$x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0$.	$x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0$.

MATRIX NOTATION

$[x_1, \ldots, x_n] \le [y_1, \ldots, y_n]$ means $x_1 \le y_1$ & ... & $x_n \le y_n$.

Let C•X = the inner product. For columns, $C \bullet X = C^T X$.

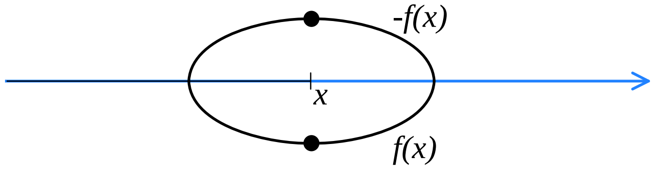
If $C = [c_1; \ldots; c_n]$, $A = [\, a_{11}, a_{12}, \ldots, a_{1n}; \ldots ; a_{m1}, \ldots, a_{mn}]$ $B = [\, b_1; \ldots; b_m]$ and $X = [\, x_1;\ x_2;\ \ldots;\ x_n]$, then the two forms can be written in matrix notation.

Standard form:	*Canonical form:*
maximize $z = C \bullet X$	maximize $z = C \bullet X$
subject to	subject to
$AX \le B$	$AX = B$
$X \ge 0$	$X \ge 0$

THEOREM. Every general linear programming problem can be rewritten in standard and in canonical form.

PROOF.

- Rewriting min in terms of max.
  x minimizes $f(x)$ iff x maximizes $-f(x)$



- Rewriting $\ge$ with $\le$. $ax \ge b$ iff $-ax \le -b$
- Rewriting = with $\le$. $ax = c$ iff $ax \le c$ & $ax \ge c$
  iff $ax \le c$ & $-ax \le -c$
- Rewriting $\le$ with = & $s \ge 0$.
  $ax \le b$ has a solution iff
  $ax + s = b$, $s \ge 0$ has a solution (let $s = |b - ax|$).

$s = |b - ax|$ is a *slack variable* - it is the distance or *slack* between ax and b.

- Rewriting $\ge$ with = & $s \ge 0$.
  $ax \ge b$ has a solution iff
  $ax - s = b$, $s \ge 0$ has a solution (let $s = |b - ax|$).