RECALL. For the canonical case, \( X \) is a solution iff \( AX = b \); it is feasible iff \( X \geq 0 \). A basic solution is feasible if all the basic variables are \( \geq 0 \) (the parameters, being 0, are automatically \( \geq 0 \)).

Different sequences of elementary row operations can produce different matrices in tableau form which have different sets of basic variables. In general (from last lecture)

**Definition.** A set of \( m \) variables is basic iff its associated set of columns is independent.

**Theorem.**

(a) A set of \( m \) variables is basic iff it is the set of basic variables for some tableau.

(b) Every set of \( m \) basic variables uniquely determines a basic solution.

(c) \( X \) is a basic solution iff \( X \) is a solution and the columns of \( A \) associated with \( X \)'s nonzero entries are independent.

**Proof.**

(a) A set of \( m \) columns is independent iff they can be transformed, using row operations, into \( m \) independent identity columns of a tableau matrix whose variables become are the tableau's basic variables.

(b) After the \( k \) parameters are set to 0, one is left with \( m \) equations in the \( m \) basic variables. Thus the solution is unique.

(c) Since the parameters are 0, any nonzero variables must be basic variables and they must have independent columns. □

**Fundamental Theorem.** For canonical problems, extreme points and feasible basic solutions are the same thing.

**Proof.** Extreme points are exactly the points with more 0's than nearby points. This is also true for feasible basic solutions.

In basic solutions, the arbitrary parameters are 0. Nearby points are obtained by varying the arbitrary parameters.

But since the parameters are all 0, any sufficiently small change to a nearby point will make one or more of them nonzero and will produce a point with fewer 0's. Hence nearby points have fewer 0's □

CONTINUING ASSUMPTION. \( AX = b, X \geq 0 \) is a canonical problem with \( n \) variables (columns), \( m \) independent equations (rows) and \( k = n - m = \) the number of parameters = the dimension of the solution space.

In a basic solution, the basic variables can be zero or nonzero but the \( k \) parameters are 0. Hence they have \( \leq m \) positive entries, \( \geq k \) 0's.

- Every set of \( m \) independent columns of \( A \) determines a set of \( m \) basic variables.
- Every set of \( m \) basic variables uniquely determines a basic solution. (After the \( k \) parameters are set to 0, one is left with \( m \) equations in the \( m \) basic variables. Thus the number of variables = the number of independent equations which implies that the solution is unique).
- Every set of \( m \) basic variables determines a general solution in which the basic variables are written in terms of the arbitrary parameters.

RECALL. Each constraint and its boundary is labeled with its slack variable.

The slack variable measures how far a point is from the constraint's boundary.

- **max** \( z = x + y \) Graph.
  
  \[ r : -x + y \leq 1 \] Give the canonical form.
  
  \[ s : x + y \leq 3 \] Find the extremes.
  
  \[ t : x - y \leq 1 \] Label them with their basic variables.

\( x, y \geq 0 \)

Canonical form: The variables are \( \{x, y, r, s, t\} \).

\[
\begin{align*}
  \text{max } z &= x + y \\
  r : -x + y + r &= 1 \\
  s : x + y + s &= 3 \\
  t : x - y + t &= 1 \\
  x, y, r, s, t &\geq 0 \end{align*}
\]

In basic solutions, the arbitrary parameters are 0. Nearby points are obtained by varying the arbitrary parameters.

But since the parameters are all 0, any sufficiently small change to a nearby point will make one or more of them nonzero and will produce a point with fewer 0's. Hence nearby points have fewer 0's □

**Question:** How do adjacent pairs differ from nonadjacent?
LEMMA. The number of basic solutions is \( \binom{n}{m} \).

PROOF. Suppose the problem has \( n \) variables (hence \( n \) columns) and \( m \) rows (assumed independent).

The number of basic solutions

\[ = \text{the number of ways of choosing } m \text{ indep. columns} \]

\[ \leq \text{the number of ways of choosing } m \text{ columns} = \binom{n}{m}. \]

Brute force method: Locate all feasible basic solutions and calculate their objective values and find largest.

This isn’t practical for problems with \( \geq 1000 \) variables since the number \( \binom{1000}{m} \) becomes astronomical.

- The unnamed numbers 6, 9 after “u” are the objective values.

\[
\begin{align*}
\text{y }&\text{ u} = x \ y \ u & (0, 3, 4) \text{ basic: } \{y, u\}, \text{ parameter: } x \\
\text{x u} &= (5, 0, 2) \text{ basic: } \{x, u\}, \text{ parameter: } y \\
\end{align*}
\]

- Suppose there are 6 feasible basic solutions as listed below with 5 variables \( x, y, z, u, v \).

The unnamed third number is the objective value which we are maximizing.

The values of the basic variables are listed; the unlisted parameters are always 0.

Adjacent vertices have been connected with lines.

The **boldface extremes and arrows** mark the path which starts at the bottom obtained and always moves to the adjacent extreme with largest objective value.

SIMPLEX METHOD. Start with a simple extreme point.

At each stage move to an adjacent extreme with a larger objective value (we will pick the one in the steepest upward direction).

When no adjacent extreme has a larger value, stop. You have a (local and \( \therefore \) absolute) maximum value.

**Theorem.** An extreme \( p \) is adjacent to an extreme \( q \) iff one can get \( q \)’s set of basic variables from \( p \)’s set by adding a new entering basic variable and deleting an old departing basic variable.

**Proof (nondegenerate case where all basics are positive).** As you move away from an extreme, you move away from one of the boundaries. The parameter for this boundary goes positive and it becomes the entering basic variable.

When you hit an adjacent boundary, the positive basic variable for that boundary goes to 0 and becomes a parameter. It is the departing basic variable.