

Math 414 Lecture 6

PC lab day Wednesday. Bring diskette.

Standard form:

maximize

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0.$$

Canonical form:

maximize

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n =$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n =$$

$$\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0.$$

ABBREVIATION. $x, y, u, v \geq 0$ means $x \geq 0, y \geq 0, u \geq 0, v \geq 0$.

Reducing unrestricted variables to nonnegatives.

$ax \leq b$ with x unrestricted has a solution iff

$a(u \geq v) \leq b$ & $u \geq 0, v \geq 0$ has a solution.

Proof. Every number x is the difference $u - v$ of two nonnegatives. $3 = (3) - (0)$, $-3 = (0) - (3)$.

Convert to standard and to canonical form.

$$\min z = x - 2y$$

$$\max -z =$$

$$\max -z =$$

with

with

with

$$x + y = 8$$

$$x - y = 4$$

$$y \geq 0$$

x arbitrary

By multiplying by -1 if necessary, we can change any canonical form into one in which the constant on the right is always ≥ 0 . E.g., $-x + y = -3$ can be replaced with $x - y = 3$.

LEMMA. There is a 1-1 onto correspondence between the set S of feasible solutions of a standard problem and set C of feasible solutions of the problem's canonical version. Every extreme point of S lifts up to an extreme point of C by adding the slack variables and every extreme point of C projects down to an extreme point of S by deleting the slack variables. Since the objective function does not involve slack variables, it has the same value on either extreme.

Standard problem

$$\max w = x + 2y$$

with

$$s: x + y = 3$$

$$x, y \geq 0$$

Canonical version

$$\max w = x + 2y$$

with

$$x + y + s = 3$$

$$x, y, s \geq 0$$

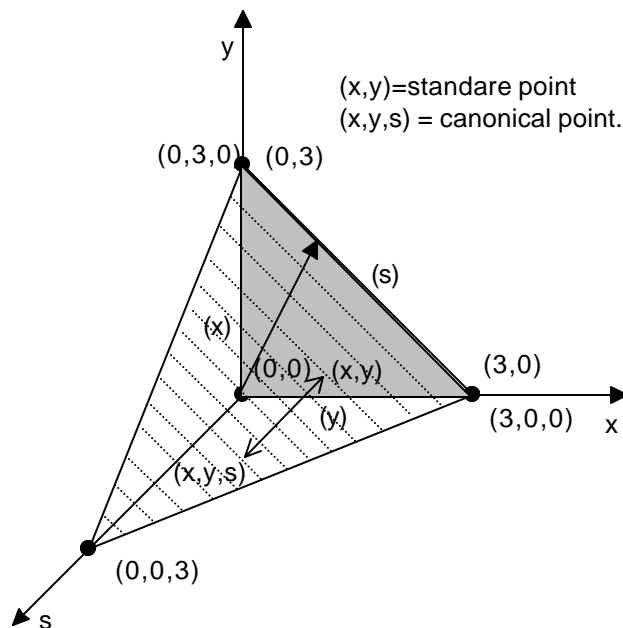
FACTS. Interior points have no zero coordinates.

Extremes have the same or more zeros than nearby points.

- A canonical coordinate is zero iff the standard point is on the boundary for that coordinate. Thus for a 2-variable

standard problem, an edge point has 1 canonical 0 coordinate; an extreme vertex point has 2 canonical 0's.

- A standard point is extreme iff it is on more boundaries than nearby points iff the canonical point has more zero coordinates than nearby points.



Standard table

x	y	w

Canonical table

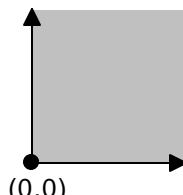
x	y	s	w

Except for the slack variable s , the tables are the same.

In the problem above, either standard or canonical, there is 1 constraint (we don't count the $x, y \geq 0$ conditions) and each extreme value has at most 1 positive entry. The rest are 0. In general we have

THEOREM. In a standard or canonical problem with m constraints, each extreme feasible solution has at most m positive entries. The rest are 0.

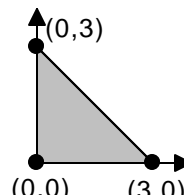
EXAMPLES.



(0,0)

no constraint

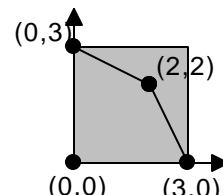
$$x, y \geq 0$$



(0,0) (3,0)

1: $x + y = 3$

$$x, y \geq 0$$



(0,0) (3,0)

1: $-x + y = 3$

2: $x - y = 3$

$$x, y \geq 0$$

Make the basic solution feasible.

$$x + y - z = -2$$

$$x, y, z \geq 0$$

$$\text{General: } x = -y + z - 2, y \text{ arb.}, z \text{ arb.}$$

Basic: $x = -2, y = 0, z = 0$

Feasible: $x = 0, y = 0, z = 2$