Math 414 Lecture 8 Exam 1: Lects. 1-8

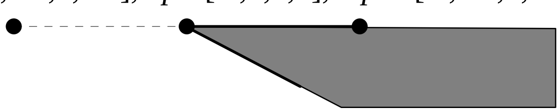
Today we consider canonical problems: $AX=b$, $X\geq 0$. We give formal proofs of theorems stated in Lecture 6.

FACT. Points sufficiently near a positive number are positive.

THEOREM. In the set S of feasible solutions, a point is extreme iff it has more 0's than nearby points.

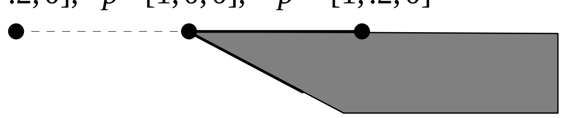
PROOF. $\Rightarrow$: Suppose p is an extreme point of S. We must show p has more 0's than nearby points. Assume for the sake of a contradiction that p' is a nearby point with the as many 0's as p. If p' is sufficiently near p, then the positive entries of p are also positive in p' by the fact above. Thus p' has 0's in the same places as p. If we let p'' be a point collinear with p and p' and sufficiently near p but on the other side from p', then p'' also has 0 and positive entries in the same places as p and also satisfies $AX=b$. The p'' is also a feasible solution. Thus p is between two other feasible points. This contradicts the assumption that p is extreme (by definition, extreme points are not between other points).

$p''=[0,1.9,0,9.2]$, $p=[0,2,0,9]$, $p'=[0,2.1,0,8.8]$



$\Leftarrow$: Suppose p has more 0's than nearby points. Assume for the sake of a contradiction that p is not extreme. Thus p is between two nearby points p' and p'' of S. p' in S and near p implies some entry is 0 in p and positive in p'. p'' on the other side of p from p' implies that entry is negative in p'' contradicting $p''\in S$.

$p''=[1,-.2,0]$, $p=[1,0,0]$, $p'=[1,.2,0]$



□

Henceforth, assume the equations linearly independent.

- Suppose the matrix form of the canonical problem is
  maximize $z=C\bullet X$ Again, $C\bullet X$ = the inner product.
  with $AX=b$ and $X\geq 0$
  where A is the $m\times n$ coefficient matrix,
  $X=[x_1;x_2;\ldots;x_n]$, $b=[b_1;b_2;\ldots;b_n]$, $C=[c_1;c_2;\ldots;c_n]$,
  n = number of variables,
  m = number of independent equations
  = number of nonzero rows in the rref
  = number of basic variables.
  = number of independent columns (row rank = column rank).
  $k=n-m$ = number of parameters.

LEMMA. Every base solution has $\leq m$ positive entries, $\geq k$ zeros.

PROOF. All the k parameters are 0. The basic variables can be zero or nonzero. □

THEOREM. k = the dimension of the set of solutions to $AX=b$.

PROOF 1. The space of all $X=[x_1,\ldots,x_n]\in \mathrm{R}^n$ has dimension n and every time we add an independent equation, the dimension is reduced by 1. After adding m equations the dimension of the set of solutions is $k=n-m$ which is the number of parameters.

PROOF 2. There are k parameters $\Rightarrow$ there are k coordinates which can vary independently $\Rightarrow$ there are k dimensions. □

RECALL. For the canonical case, X is a *solution* iff $AX=b$; it is *feasible* iff $X\geq 0$.

A basic solution is feasible if all the basic variables are ≥ 0 (the parameters, being 0, are automatically ≥ 0).

Different sequences of elementary row operations can produce different matrices in tableau form which have different sets of basic variables. When is a set of variables the set of basic variables of some tableau?

THEOREM.

(a) A set of m variables is *basic* iff its associated set of columns is independent.

(b) Every set of m basic variables uniquely determines a basic solution.

(c) X is a basic solution iff X is a solution and the columns of A associated with X's nonzero entries are independent.

PROOF.

(a) A set of m columns is independent iff they can be transformed, using row operations, into m independent identity columns of a tableau matrix whose variables become are the tableau's basic variables.

(b) After the k parameters are set to 0, one is left with m equations in the m basic variables. Thus the solution is unique.

(c) Since the parameters are 0, any nonzero variables must be basic variables and they must have independent columns. □

FUNDAMENTAL THEOREM. For cannonical problems, extreme points and feasible basic solutions are the same thing.

PROOF. Extreme points are exactly the points with more 0's than nearby points. This is also true for feasible basic solutions.

In basic solutions, the arbitrary parameters are 0.

Nearby points are obtained by varying the arbitrary parameters.

But since the parameters are all 0, any sufficiently small change to a nearby point will make one or more of them nonzero and will produce a point with fewer 0's.

Hence nearby points have fewer 0's □