Math 414 Lecture 10

For today, assume all basic variables are positive. In Lecture 12 we consider the degenerate case where a basic variable is 0.

CONTINUING CANONICAL EXAMPLE. Max $z = 2x + 3y$.

with $x - 2y + v \quad = 1$

$3x + y + \quad u \quad = 4$

$x + y \quad + \quad w \ = \ 2, \qquad x, y, u, v, w \geq 0$

DEFINITION. An *augmented matrix* for a canonical problem is the augmented matrix for the system of equations plus an added *objective row* at the bottom for the objective function.

■ Since $z = 2x + 3y$, $z - 2x - 3y = 0$. The objective row is $\langle z, -2, -3, ..., 1, 0\rangle$.

	x	y	u	v	w	b = constants column	
v	1	-2	0	1	0	1	1/-2 θ ratios
u	3	1	1	0	0	4	4/1
w	1	1	0	0	1	2	2/1
z	-2	-3	0	0	0	0	← objective row

- The initial objective row entries are the negatives of z's coefficients.
- In general, the objective row is the objective function written in terms of the current parameters.
- The bottom right corner is the current objective value.
- Label each row with its basic variable. Label the first row *"v"* since v is the basic variable whose pivot point is in the first row. u's pivot lies in row 2, hence row 2 is labeled u.
- The value of a basic variable = the constant column entry for its row. Thus in any *basic solution*, the values of the basic variables and the objective function are listed in the constant column.

RULE 1. EXCEPT FOR THE OBJECTIVE ROW, THE CONSTANT COLUMN ENTRIES MUST ALWAYS BE NONNEGATIVE. Reason: each solution must be feasible.

RULE 2. PIVOT ONLY ON POSITIVE COEFFICIENTS. Pivoting on a negative coefficient will produce a negative constant column entry.

RULE 3. THE PIVOT COLUMN'S VARIABLE ENTERS; THE PIVOT ROW'S PREVIOUS VARIABLE DEPARTS.

Simplex method *Start with an intial extreme point. At each stage move to the adjacent extreme with a larger objective value. When no adjacent extreme has a larger value you have a maximum.*

To start, pick a set of basic variables with a feasible solution.

■ Clearly the u, v and w columns are independent, so let $\{u, v, w\}$ be the initial set of basic variables.

Loop: Write the basic variables and the objective function in terms of the parameters and a constant. To do this

Pivot on the columns of the basic variables. Label the rows with the basic variable which has a 1 (i.e., pivot) in its row.

■ This matrix is already in tableau form for u, v, w so we don't need to pivot. We have labeled the rows v, u, w.

Solving for v, u, w gives $v = 1 - x + 2y$, $u = 4 - 3x - y$, $w = -x - y$.

Solving for z gives $z = 0 + 2x + 3y$.

■ In the basic solution, the parameters are 0, thus $v = 1$, $u = 4$, $w = 2$ and $z = 0$ which is exactly the last column $\langle 1; 4; 2; 0\rangle$ of the matrix. The last column gives the current basic solution plus it's objective value.

Now we want to find an adjacent extreme with a larger objective value. To do this we must select a new entering basic variable and then a departing basic variable.

- Increasing a parameter with a positive objective coefficient increases the objective function. To get the largest increase, pick the parameter with the largest positive objective coeffficient and thus the most *negative* objective row entry.

■ Since $z = 2x + 3y$, choose y for the largest increase in z. In the objective row, this reads $z - 2x - 3y = 0$.

Pick a variable with the most negative entry in the objective row to be the entering basic variable.

■ Since the bottom row is $\langle -2, -3, 0, 0, 0, 0\rangle$, y is the entering variable.

■ If we had $z = -4x - 7y - w$, then increasing x, y or w decreases z and so all adjacent extremes have a lower value. Thus we would be at a maximum. In this case the objective row would be $z + 4x + 7y + w = 0$.

If no objective row coefficient is negative, stop; you have an optimal solution.

Given our entering variable, we must find the departing variable.

■ From the tableau we get,

$x - 2y + v = 1$, $3x + 1y + u = 4$ and $1x + 1y + w = 2$

Of the parameters x and y, y has been chosen to be the new basic and hence x continues as a parameter. In the basic solution, it is 0. Hence equations simplify to,

$-2y + v = 1$ and $1y + u = 4$ and $y + w = 2$. Thus

$v = 0$ iff $-2y = 1$ iff $y = -1/2$ and

$u = 0$ iff $1y = 4$ iff $y = 4/1$ and

$w = 0$ iff $y = 2/1$ -- note constant column 2 divided by coefficient 1.

- Note that the ratios (called θ ratios) on the right consist of a constant column entry over a coefficient matrix entry.

$y = -1/2$ is impossible since y must be ≥ 0, thus v can't be the departing variable. If a coefficient in the entering variable's column is < 0, then the basic variable for that row cannot depart. This goes in the wrong direction.

Thus either u or w is the departing basic variable. Since $2/1 < 4/1$, w goes to 0 before u does. Thus w is the departing variable.

- If no coefficient in the entering variable's column is positive, then no basic variable ever goes to zero. Hence the entering variable and thus also the objective function can increase forever. In this case, the region is unbounded and there is no maximum.

LEMMA. Suppose u is the tableau parameter choosen to be the entering basic variable. (a) The departing basic variable which becomes a parameter in the adjacent extreme = the first basic variable to become 0 as u increases = the basic variable with the smallest constant/(positive-coefficient) ratio. (b) If all constant/(positive-coefficient) ratios are negative, the region is unbounded and there is no maximum.

Calculate the ratios (called q ratios) of the constant column entries over the coefficients of positive value in the entering variable's column (ignore the objective row). Pick the basic variable whose row has the smallest q ratio to be the departing variable.

If the entering variable's column has no positive coefficients, stop; *the region is unbounded in the objective direction*, there is no maximum.

Otherwise, repeat the loop with the new set of basic variables. Repeating the loop pivots on the new basic variables and gives a new basic solution with a higher objective value. Thus repeating the loop gives higher and higher extremes until we find the optimal solution or find that there is no optimal solution due to the region being unbounded in the objective direction.